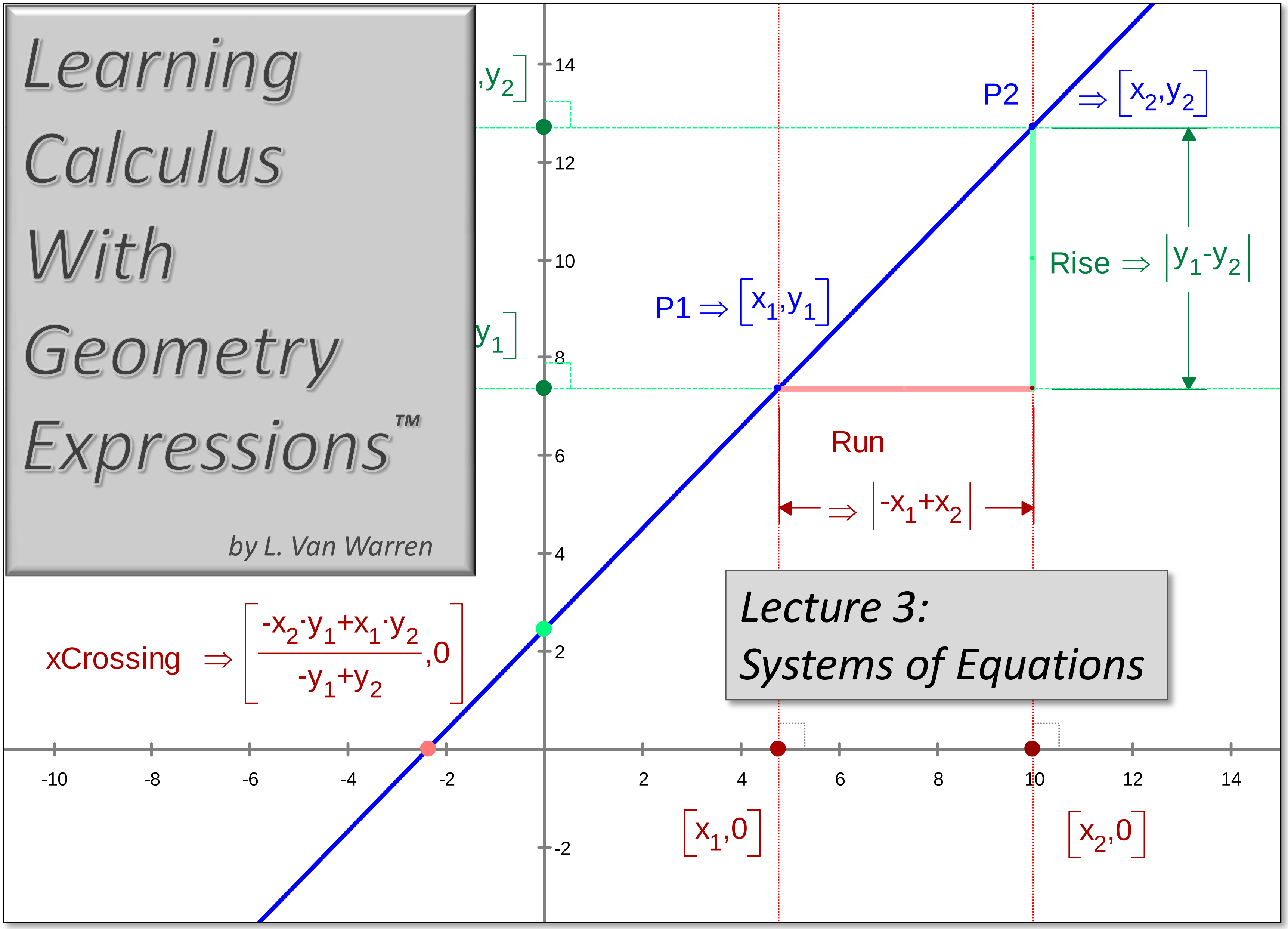
Learning Calculus With Geometry Expressions™

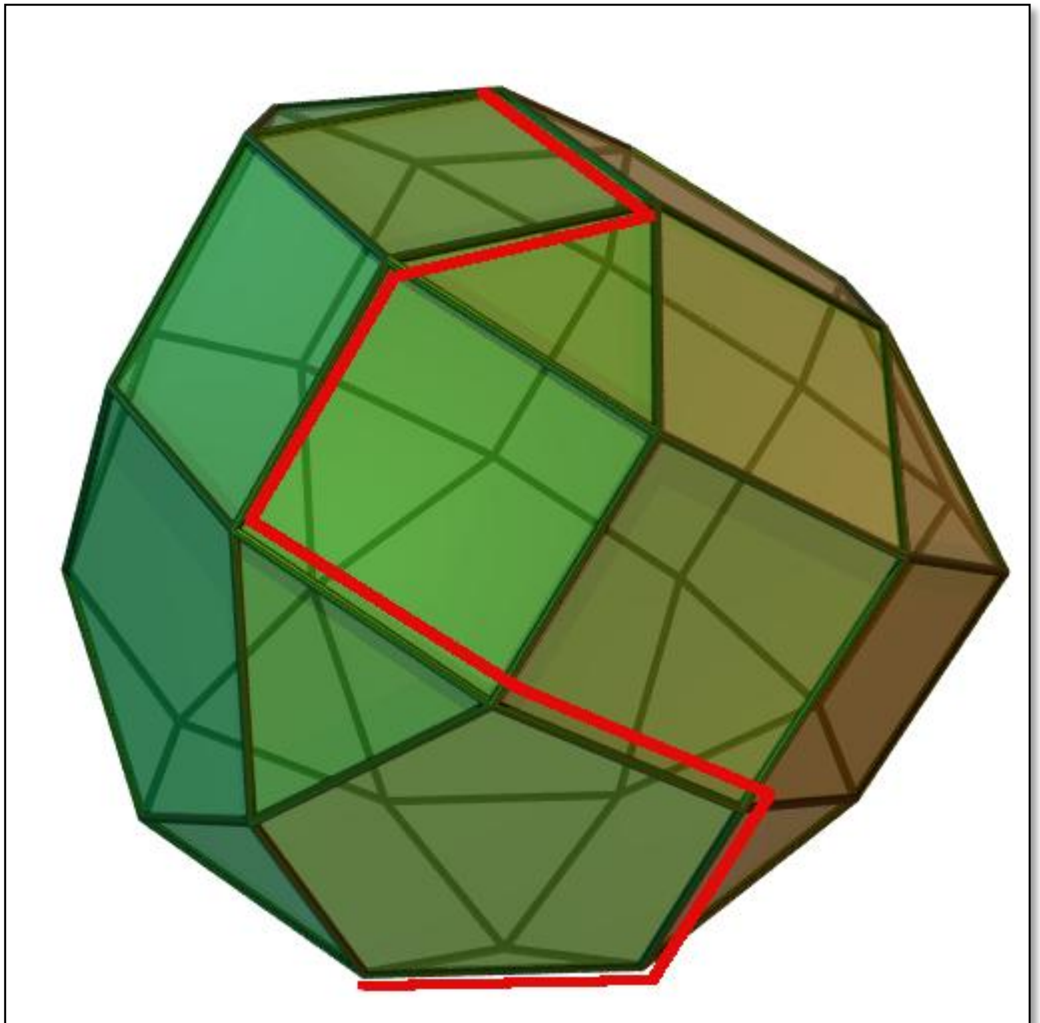
by L. Van Warren

Lecture 3: Systems of Equations



Chapter 1: Functions and Equations

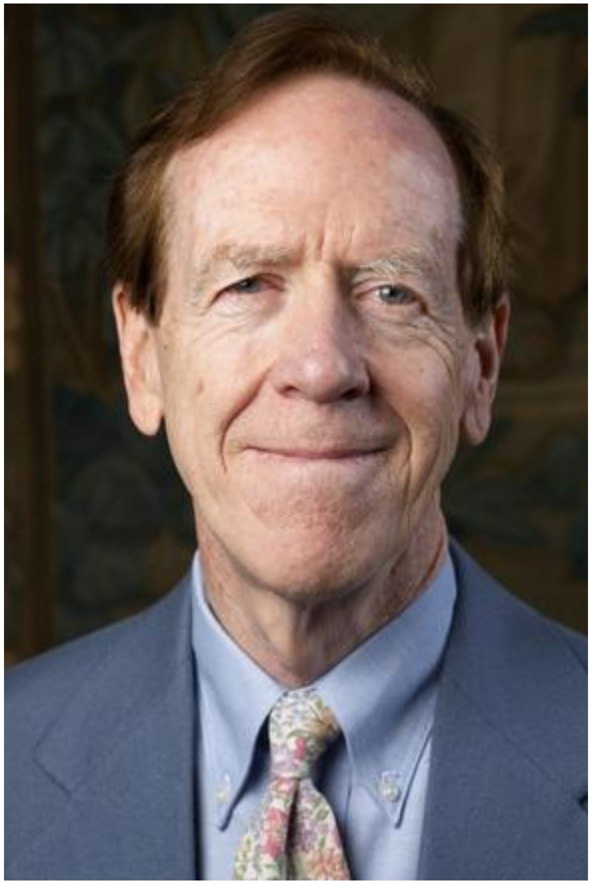
Lecture	***Topic***
0	*Geometry Expressions™ Warm-up*
1	*Explicit, Implicit and Parametric Equations*
2	*A Short Atlas of Curves*
3	***Systems of Equations***
4	*Invertibility, Uniqueness and Closure*



- *Polytope of Linear Equations from Simplex Method*

Lecture 3: Systems of Equations

INSPIRATION



Gilbert Strang, Ph.D. is an MIT professor of mathematics. He has written many excellent books in the field of computational mathematics.

One magnificent example is Linear Algebra and Applications. In this book Professor Strang explains the theory necessary for solving large systems of simultaneous equations. He discusses direct methods, such as Gaussian Elimination which produce exact numerical solutions. These require more computer time than iterative methods such as Successive Over-Relaxation (SOR) which produce answers more quickly, albeit more approximately, at first.

The Strang online linear algebra lectures are renowned and can be found at MIT's Open Courseware website.

Systems Of Equations

Mathematicians, Scientists and Engineers enjoy the illusion of being able to predict the future. When things are simple, prediction of how a system behaves is indeed possible. When explicit solutions exist, the ***"solution"*** is found by writing equation(s) that govern the relationships between key system variables or parameters.

One is often interested in evaluating the solution at specific locations or moments in time. In that case, ***"Finding a solution"*** becomes synonymous with ***"root finding"*** and root finding with ***intersection.*** The word ***"find"*** implies that a search is necessary. Searching is often required in the implicit and parametric cases.

> ***Solve*** is a euphemism meaning, "***to intersect***".
> ***Root*** is another word for ***"intersection"***.

ONE EQUATION, ONE UNKNOWN

We can write the equation:

$$\boxed{y = 2 \cdot x + 4}$$

If we want the value of x that makes $y = 0$, we say we have one equation in one unknown. The place where $y = 0$ is a special place… it is the x-axis!

To find this value of x, we have to solve or rearrange the equation so that x is in terms of y. So our "explicit" equation really isn't explicit anymore! If we didn't rearrange we'd have a search; Try many different x*'s* to find the one that makes $y = 0$. Fortunately, with linear equations, this solution or rearrangement is always possible. In this case it would be:

$$\boxed{x = \frac{1}{2} y - 2}$$

To find the value of x that makes $y = 0$, we set y to zero in the equation above. The value of x that makes $y = 0$ is called an ***x-axis crossing*** or ***root***. In this case the value is $x = -2$, so we say the equation has one root and that the root is -2. This process enables us to solve one equation in one unknown. We are intersecting the line given above with the x-axis (whose equation is y=0), to obtain the root or solution.

More Than One Path

When we wrote the equation: $\boxed{y = 2 \cdot x + 4}$

We never said how we obtained it. There are two ways we could have:

1) Slope-intercept form: $\boxed{y = m \cdot x + b}$

Given two points, we computed the slope of $\boldsymbol{m = 2}$, and the $\boldsymbol{y}$-intercept of $\boldsymbol{b = 4}$.

2) Point-slope form: $\boxed{y - y_1 = m(x - x_1)}$

Given one point and a slope, we first compute $\boldsymbol{m}$, then the $\boldsymbol{y}$-intercept.

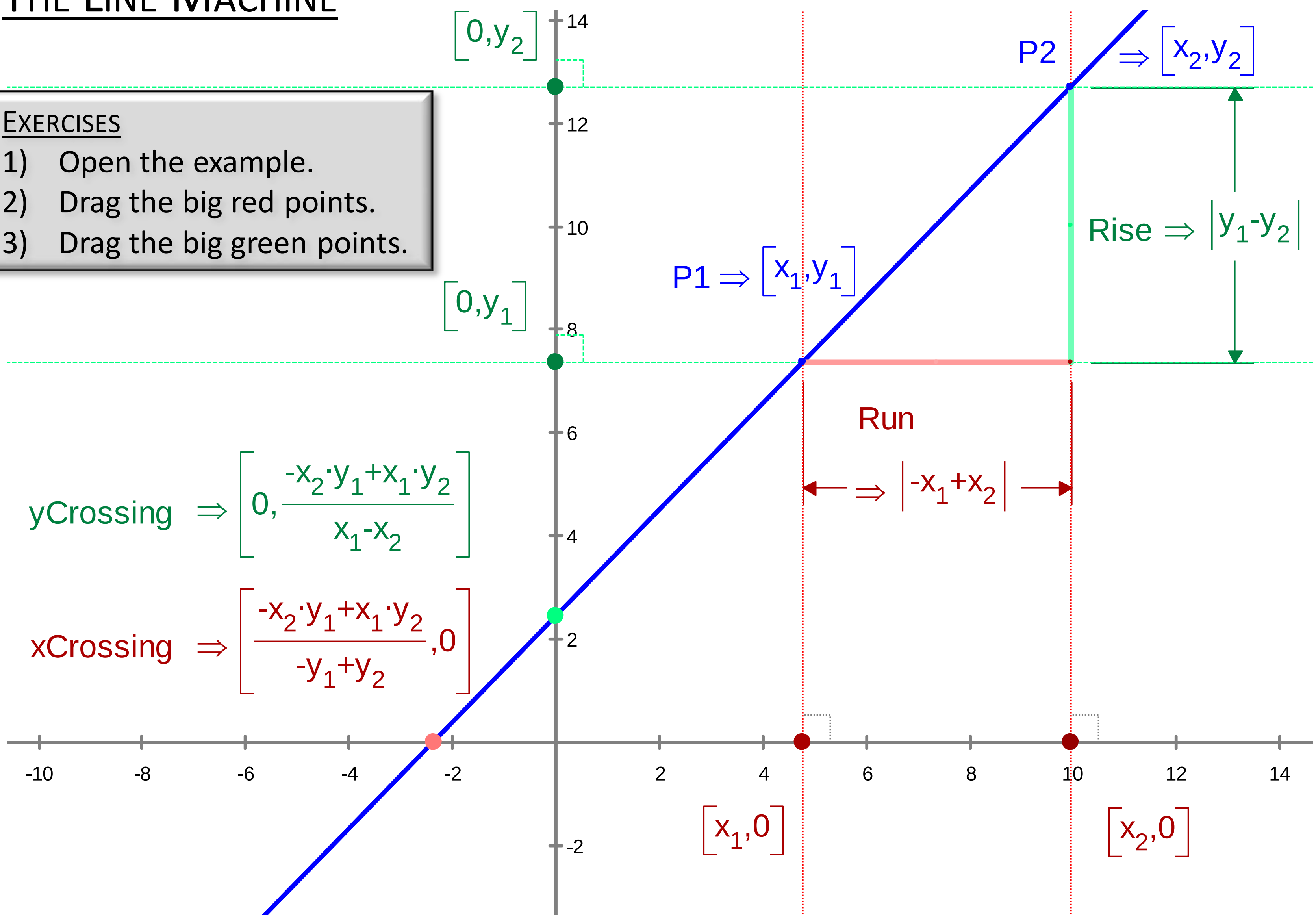
The Geometry Expressions™ file on the next page demonstrates the former. Try it!

The Line Machine

Lecture03-OneEquationOneUnknown.gx

Exercises

1) Open the example.
2) Drag the big red points.
3) Drag the big green points.



TWO EQUATIONS, TWO UNKNOWNS

When we had one equation and one unknown, we were intersecting two lines, but one line, the x-axis, was degenerate, i.e. its equation was $y = 0$. Imagine another line springing from the x-axis, that has its own equation, slope and y-intercept. The intersection still occurs, but not at $y = 0$. The intersection now happens at an (x, y) that ***simultaneously*** satisfies both equations, thus the oft-heard term. In the figure on the next page this intersection is labeled P, for (Xintersection, Yintersection).

We call x and y, scalars, since they are single-valued variables, and this allows me to repeat myself. We have two equations and two scalar unknowns, x, and y. As you can clearly see, these two equations have one and only one simultaneous solution.

Using Geometry Expressions™ we found the symbolic and numerical solution to these two equations. In effect the problem has been solved for all time, because we can evaluate the symbolic solution for any case we want. This is VERY useful. As Prof. Will Worley of the University of Illinois pointed out, “It is often as easy to solve an entire class of problems, as it is to solve a specific problem.”

Now we can go home and never worry about linear equations again. Well almost…

Two Equations, Two Unknowns

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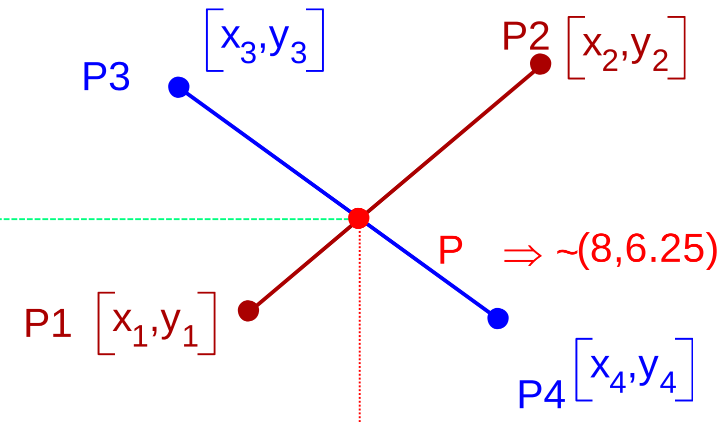
Two Equations, Two Unknowns

1) Open the example.
2) Drag the dark red and blue points.



Yintersection =

$$\Rightarrow \left| \frac{-[-x_2 \cdot y_1 + x_1 \cdot y_2] \cdot [y_3 - y_4] + [y_1 - y_2] \cdot [-x_4 \cdot y_3 + x_3 \cdot y_4]}{-[-x_3 + x_4] \cdot [y_1 - y_2] + [-x_1 + x_2] \cdot [y_3 - y_4]} \right|$$

Xintersection =

$$\Rightarrow \left| \frac{[-x_3 + x_4] \cdot [-x_2 \cdot y_1 + x_1 \cdot y_2] + [x_1 - x_2] \cdot [-x_4 \cdot y_3 + x_3 \cdot y_4]}{-[-x_3 + x_4] \cdot [y_1 - y_2] + [-x_1 + x_2] \cdot [y_3 - y_4]} \right|$$

Calculating Intersection Values

1) Draw two line segments.
2) Constrain their coordinates.
3) Select both lines.
4) Click Construct→Intersection.
5) Click Calculate→Symbolic→(x,y)

When Things Go Wrong

Three situations hinder us from solving the intersection of ANY two lines:

Case 1: Two Parallel Lines:
Two parallel lines with the same slope never intersect and thus have no simultaneous solution. Convince yourself of this by moving points P1 and P2 until the two lines are parallel.

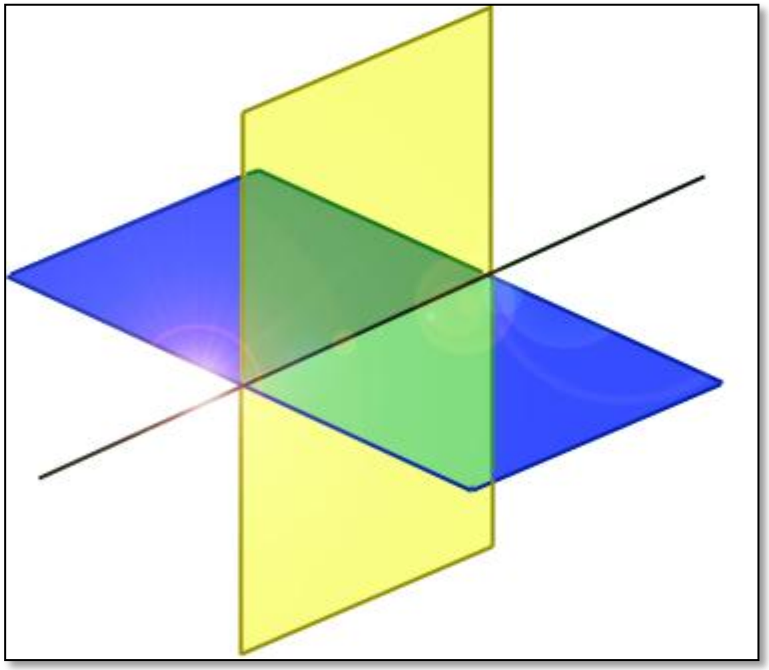
Case 2: Two Coincident Lines:
This is a special case of case 1. The lines not only have the same slope they have the same y-intercept, in other words they are linear multiples of each other: $Ax + By = C$ can be multiplied by any number. Try moving points P1 and P2 so they are coincident with points P3 and P4 to see this.

Case 3: Nearly Parallel Lines:
As the lines approach being parallel the intersection point quickly goes to infinity. This creates numbers too large to represent conveniently. Computers have finite word lengths and the result is numerical overflow. In this case we say the equations are ***numerically unstable***. One solution to this is to solve the problem symbolically and ***take the limit***. We will be looking at that in the next chapter.

Three Equations In Three Unknowns

The conversation we had for pairs of lines, can be extended to sets of planes. Imagine we intersected two planes in three-dimensional space. The intersection forms a line seen in the figure. If we brought in a third plane, at a right angle to the other two planes, the three planes would intersect at a single point.
We would have three equations in three unknowns like so:

$$\begin{aligned} a_0 \cdot x + b_0 \cdot y + c_0 \cdot z &= d_0 \\ a_1 \cdot x + b_1 \cdot y + c_1 \cdot z &= d_1 \\ a_2 \cdot x + b_2 \cdot y + c_2 \cdot z &= d_2 \end{aligned}$$



Incidentally the coefficients ***a***, ***b*** and ***c*** represent the components of a vector pointing away from a given plane… but that's another story. In this case we can solve ***linear systems*** using Cramer's Rule. When we have more unknowns Cramer's Rule is too slow, so techniques described by Gilbert Strang and others are used to speed up the process. For now we will just stick with equations that lie in a ***single*** plane. There is plenty to do there!

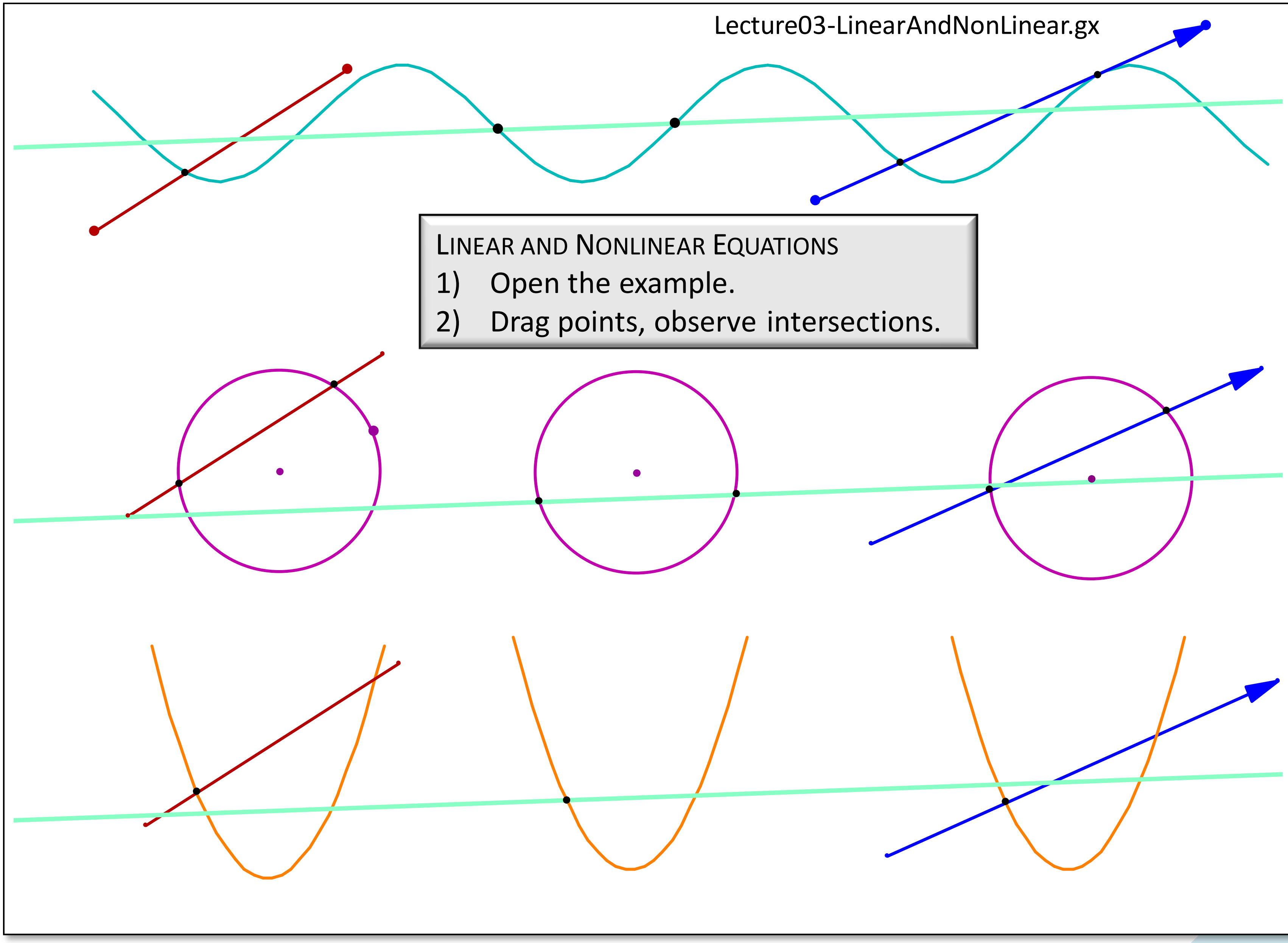
LINEAR AND NONLINEAR SYSTEMS OF EQUATIONS

We have discussed three representations of equations. To “solve” or intersect all the possible representations with each other we have to take the Cartesian product of the set {*explicit, implicit, parametric*} with itself. The nine possible combinations are ***enumerated*** (counted up) in the table below.

For the nonlinear case, we have to take each problem on a case-by-case basis. Indeed a mountain of mathematicians have devoted their lives to addressing nonlinear problems and the families of possible intersections that exist.

Maximum Number of Intersection Points	$y=mx+b$	$ax+by-c=0$	$(at+a_0, bt+b_0)$
$y = \sin(x)$	∞	∞	∞
$x^2 + y^2 - r^2 = 0$	2	2	2
(t, t^2)	2	2	2

Try the example on the next page, three linear equations intersected with three nonlinear equations.



LINEAR AND NONLINEAR EQUATIONS

1) Open the example.
2) Drag points, observe intersections.

End