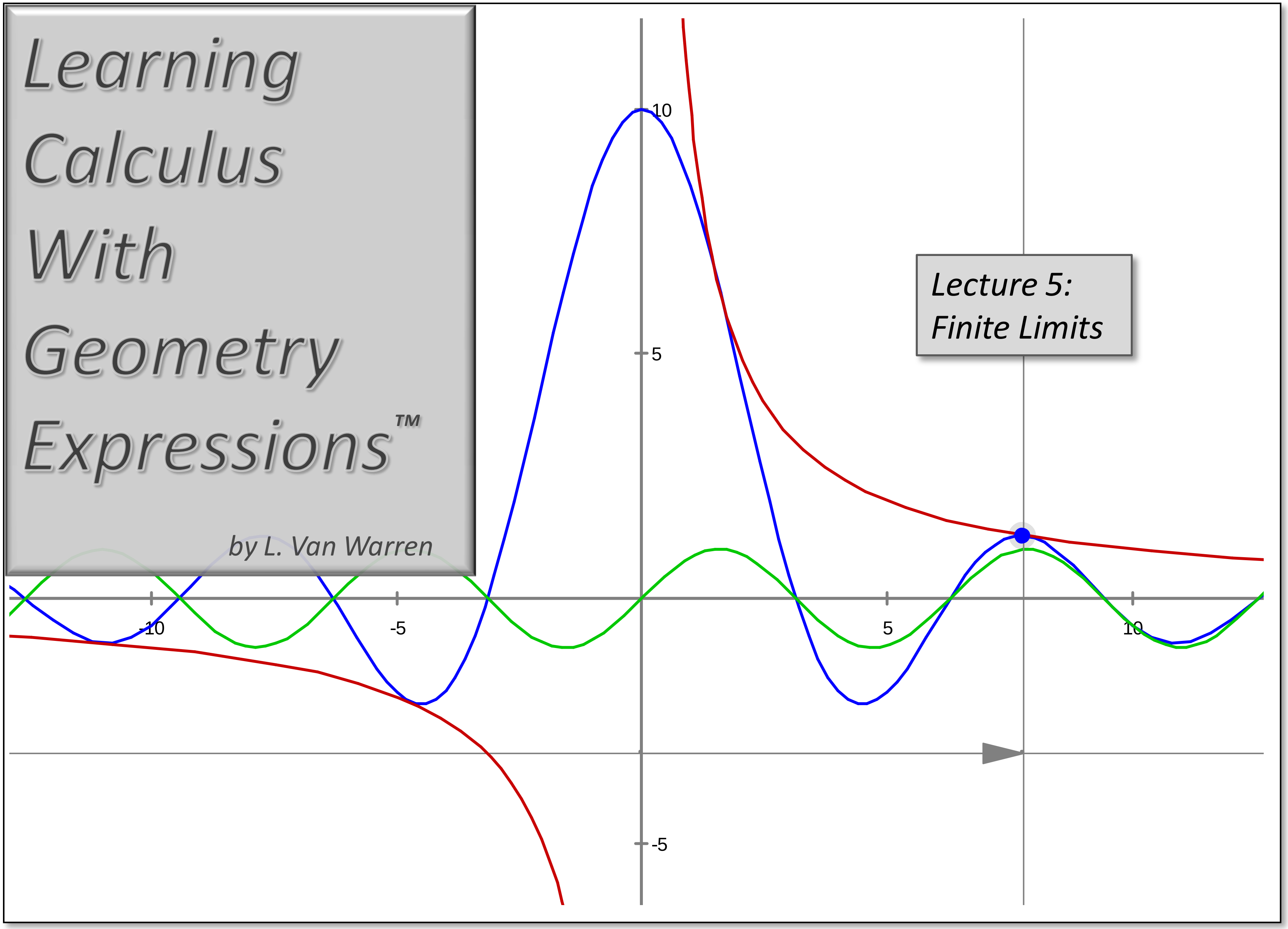
Learning Calculus With Geometry Expressions™

by L. Van Warren

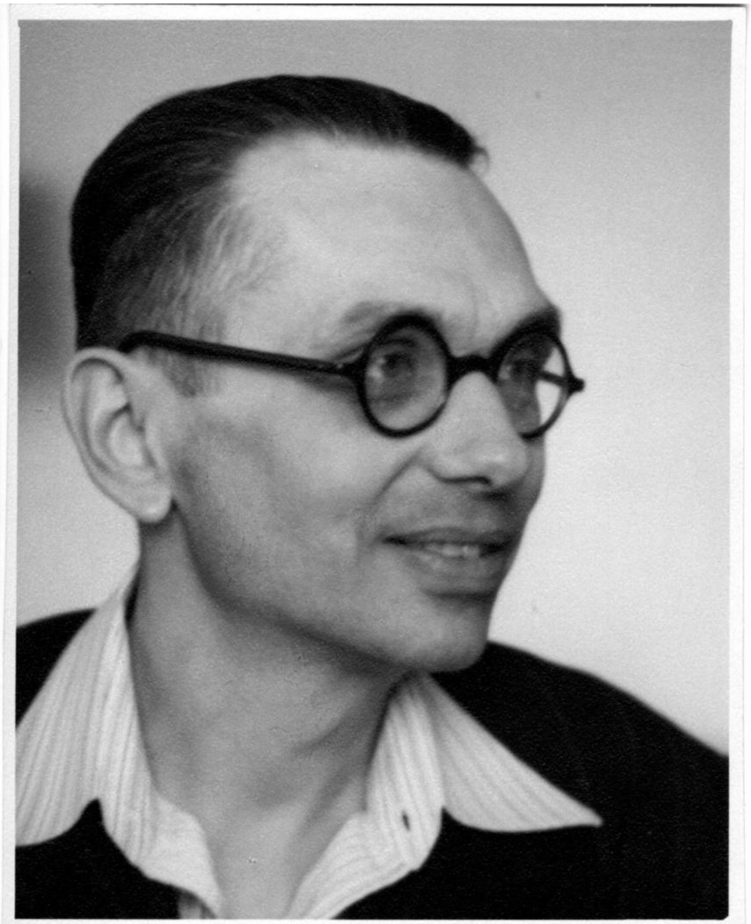
Lecture 5: Finite Limits



Chapter 2: Limits

Lecture	Topic
5	**Finite Limits**
6	Infinite Limits
7	Continuity
8	Discontinuity
9	Precise Definition of The Limit

Inspiration



Photographer unknown. Courtesy of the Archives of the Institute for Advanced Study, Princeton, NJ, USA.

Kurt Gödel, Ph. D.

Articulated *Incompleteness* and *Undecideability*. Ideas like *Closure*, only bigger.

Albert Einstein used to return to Princeton simply for the pleasure of "walking with Gödel in the garden."

Gödel upset the applecart, disrupting Bertrand Russell's and Alfred Whitehead's objective of codifying mathematical reasoning into one great self-consistent system. He did this by proving that in order for a system to be self-consistent one must first go outside the system. Why? One must establish the consistency of the rules of the original system before building on it. Thus an ever widening scope of systems must be examined and this process never terminates.

No limits!

A Free Computer Algebra Tool

Maxima –

A free download with rigorous tools for finding limits.

Limits of A Function

When taking the limit of a function f(x), we write:

$$\lim_{x \to a} f(x) = f(a)$$

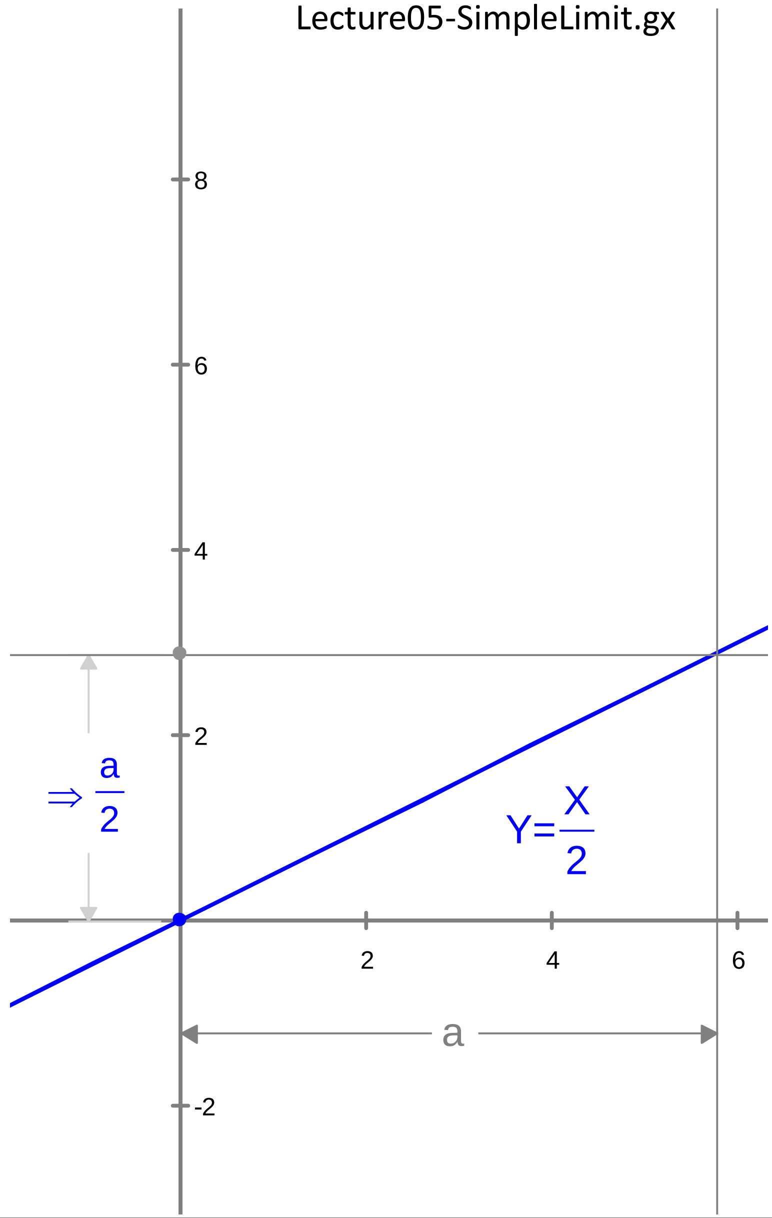
and we say, "The limit of f(x) as x goes to a is f(a)"

which means:

"Find the value of the function f(x) as x goes to a".

In this case f(x) = x/2

So the limit of x/2 as x→a is simply a/2.

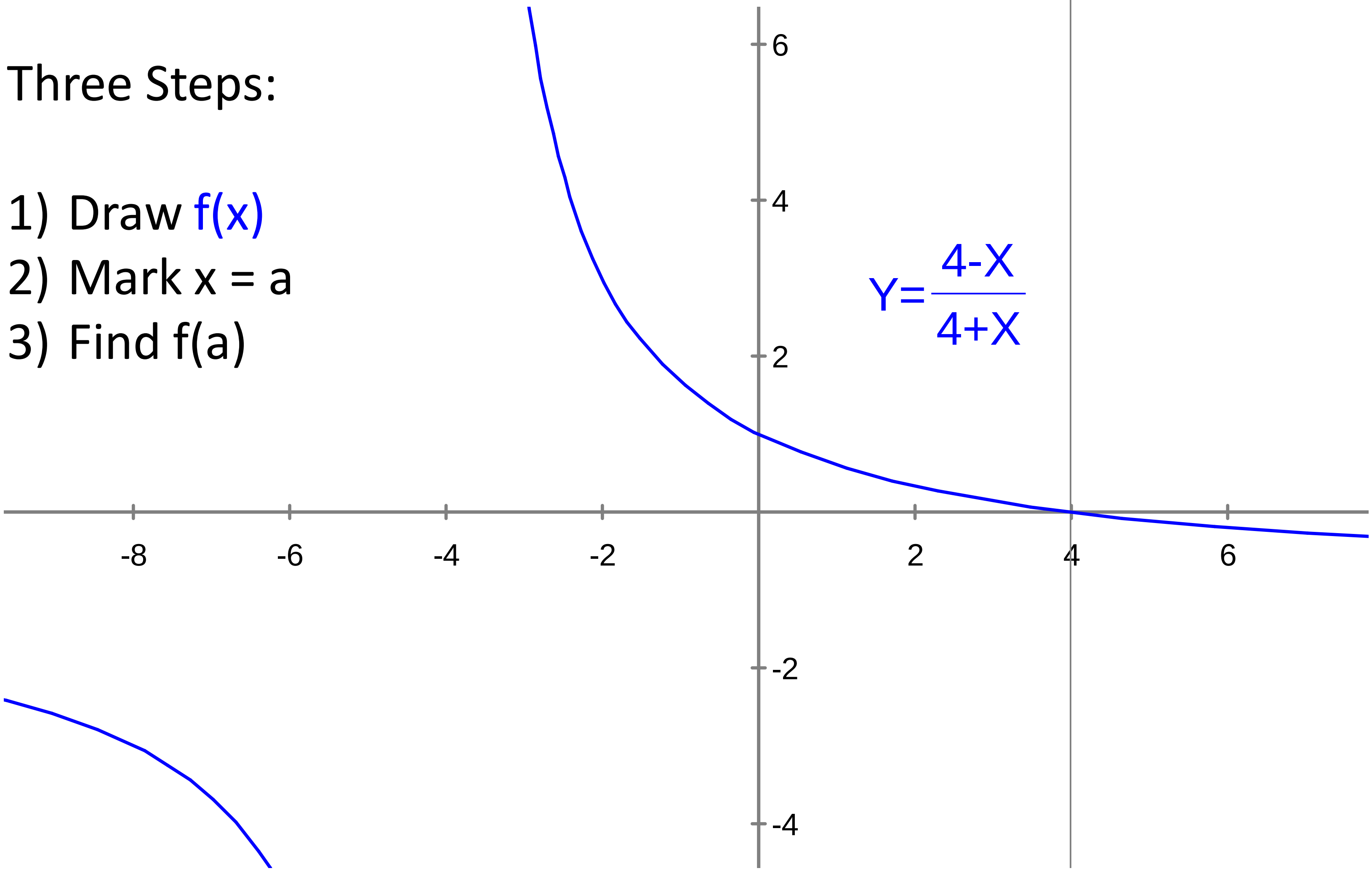


Lecture05-LimitSteps.gx

Taking the Limit

Three Steps:

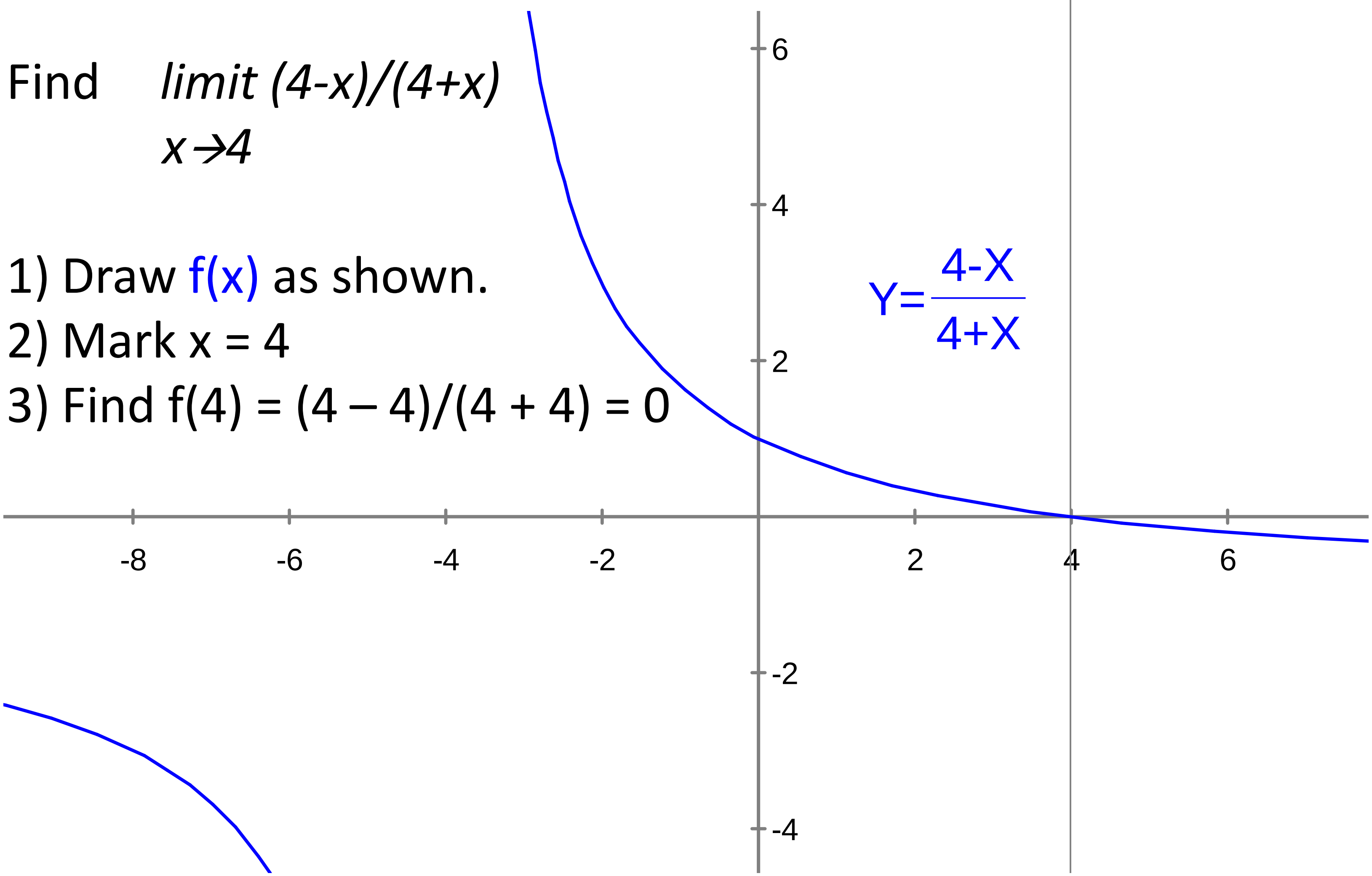
1) Draw f(x)
2) Mark x = a
3) Find f(a)



Taking the Limit Example

Find $\lim_{x \to 4} (4-x)/(4+x)$

1) Draw f(x) as shown.
2) Mark x = 4
3) Find f(4) = (4 – 4)/(4 + 4) = 0



Limits of A Function - Notation

We can also write the expression:

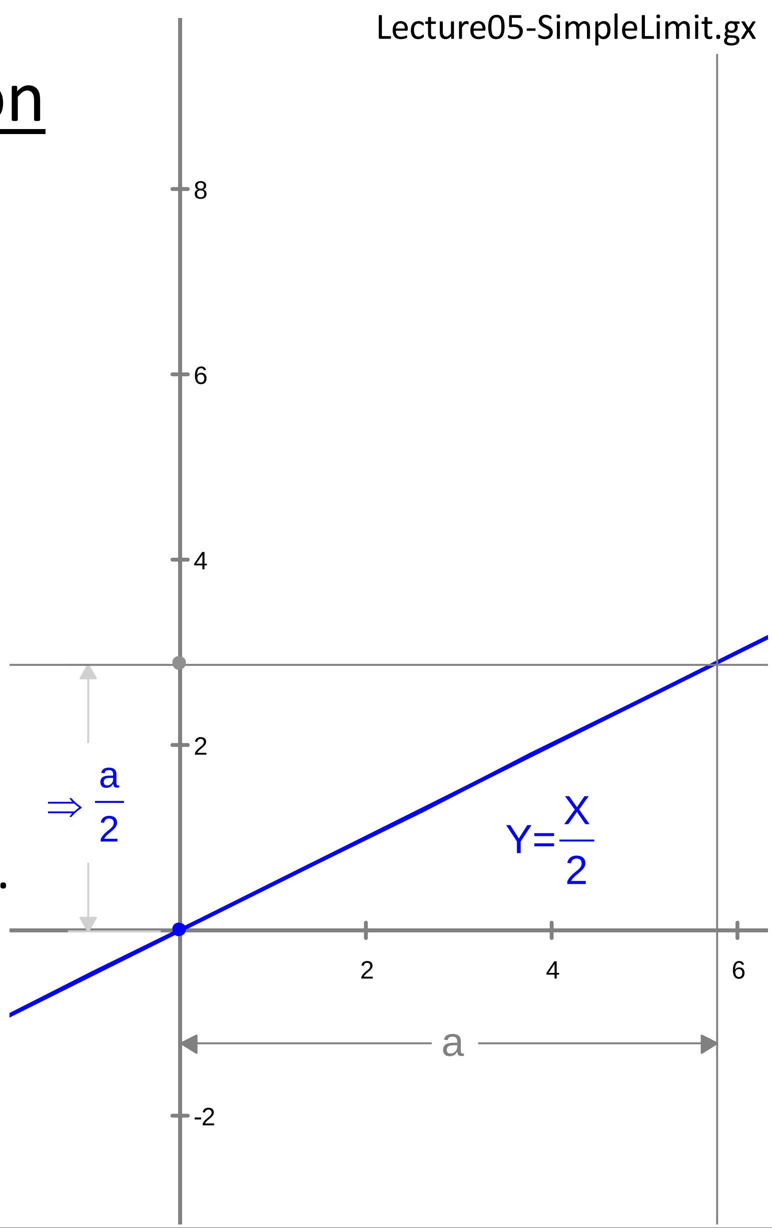
$$\lim_{x \to a} f(x) = f(a)$$

In functional notation as:

limit(f(x), x, a)

Imagine the "x, a" arguments mean x→a.

Thus limit () is a function that takes a function as input and produces another as output.



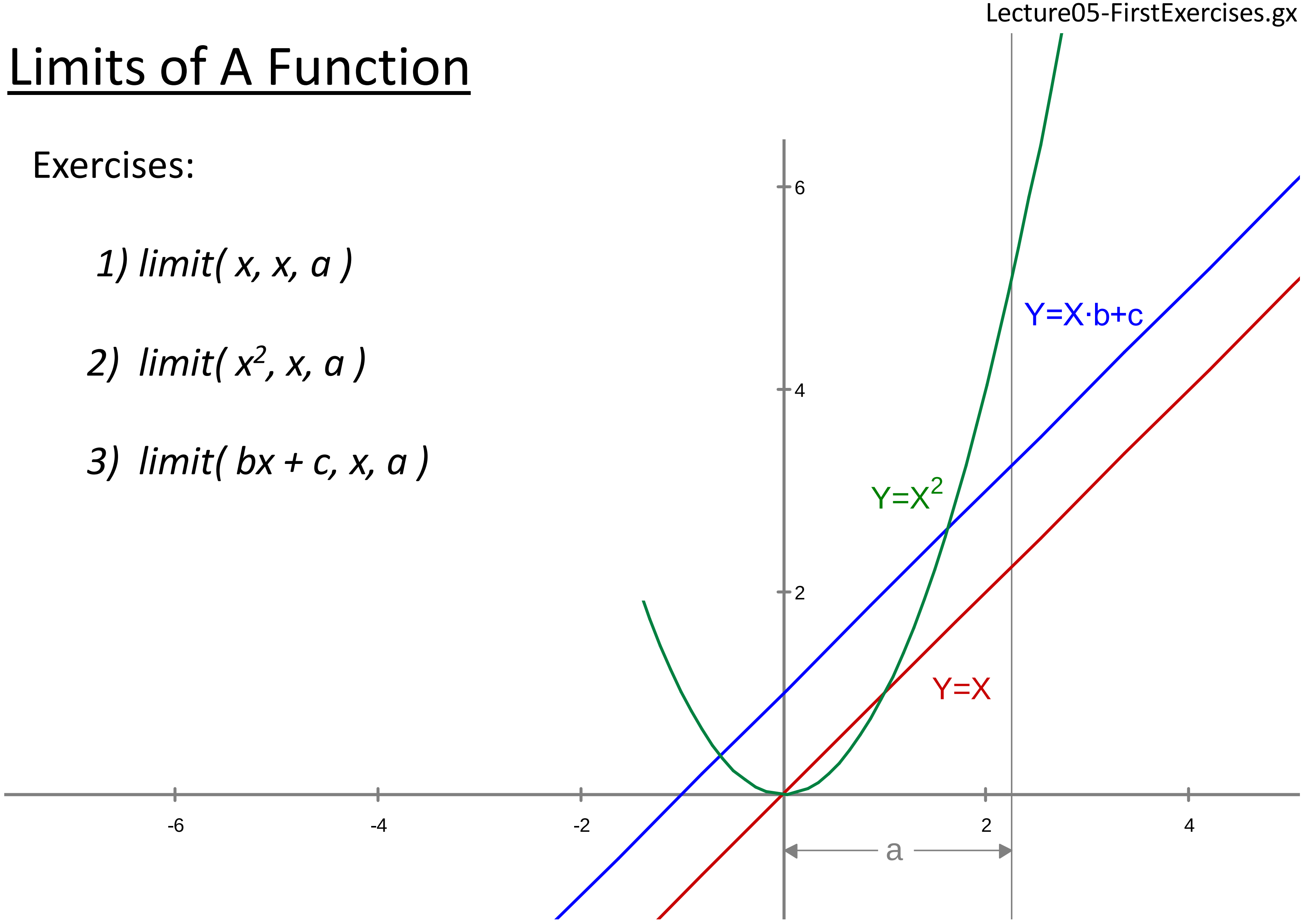
Limits of A Function

Exercises:

1) *limit(x, x, a)*

2) *limit(x^2, x, a)*

3) *limit(bx + c, x, a)*



Lecture05-FIrstExercises.gx

Limits of A Function

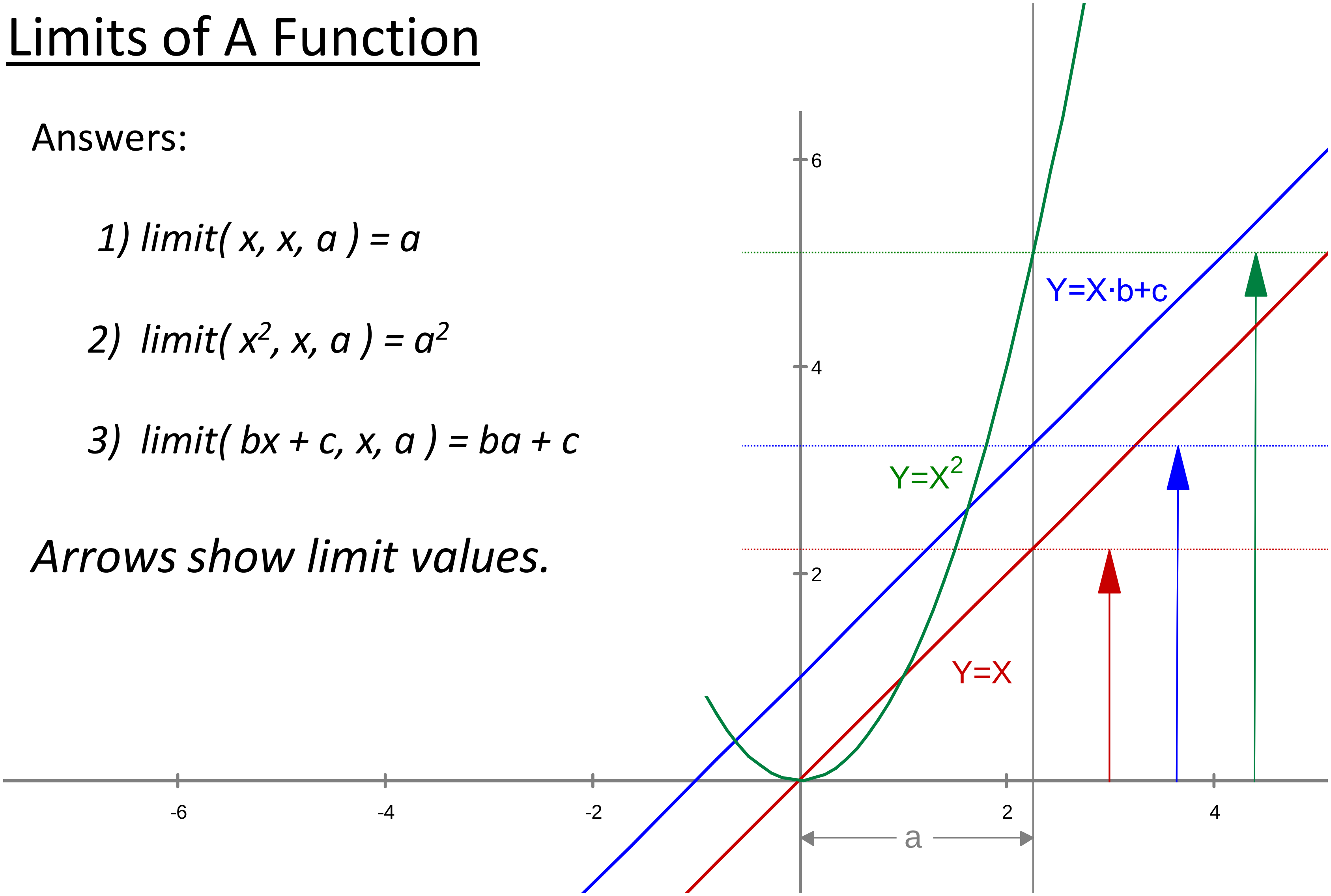
Answers:

1) *limit(x, x, a) = a*

2) *limit(x^2, x, a) = a^2*

3) *limit(bx + c, x, a) = ba + c*

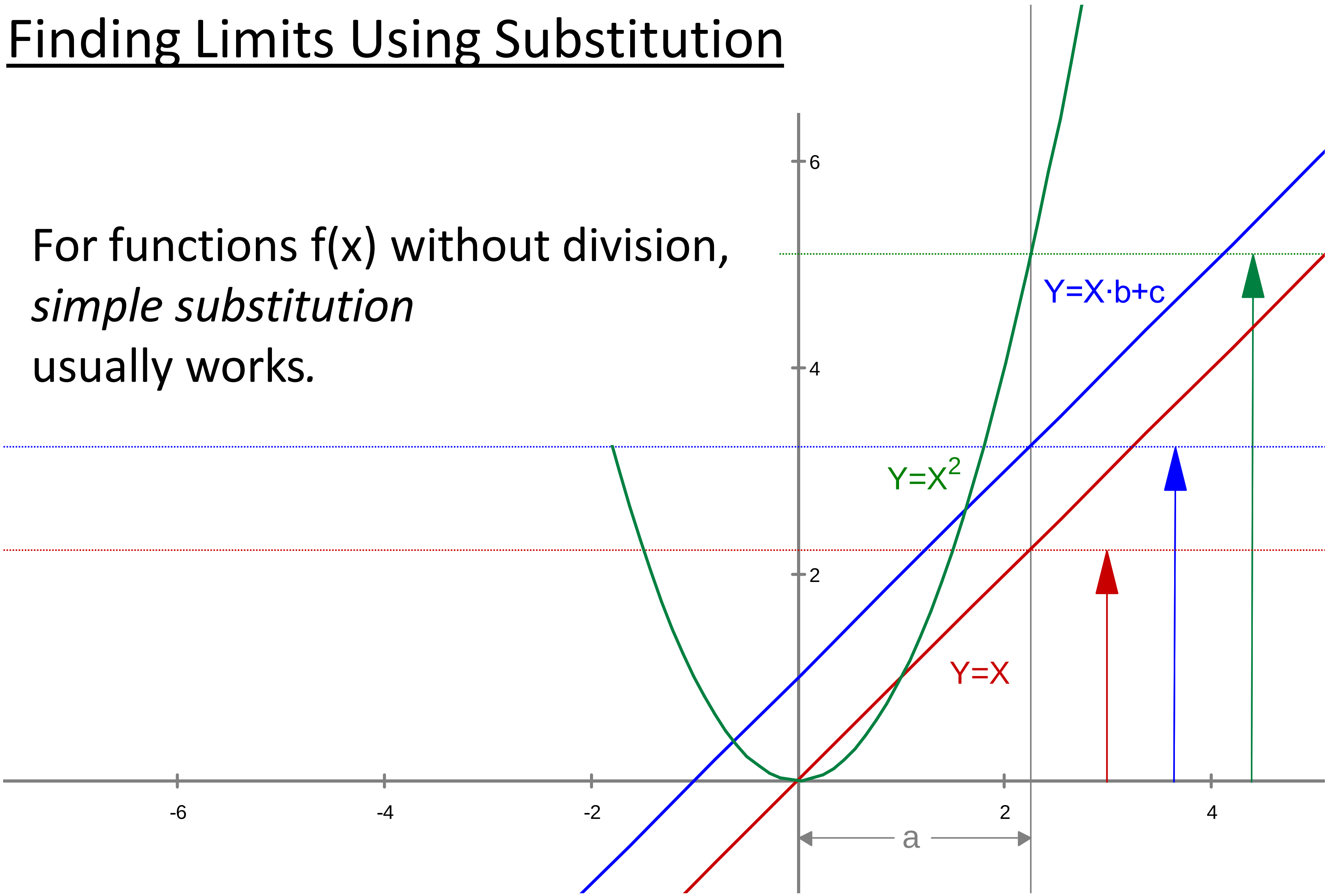
Arrows show limit values.



Lecture05-FIrstExercises.gx

Finding Limits Using Substitution

For functions f(x) without division,
simple substitution
usually works.

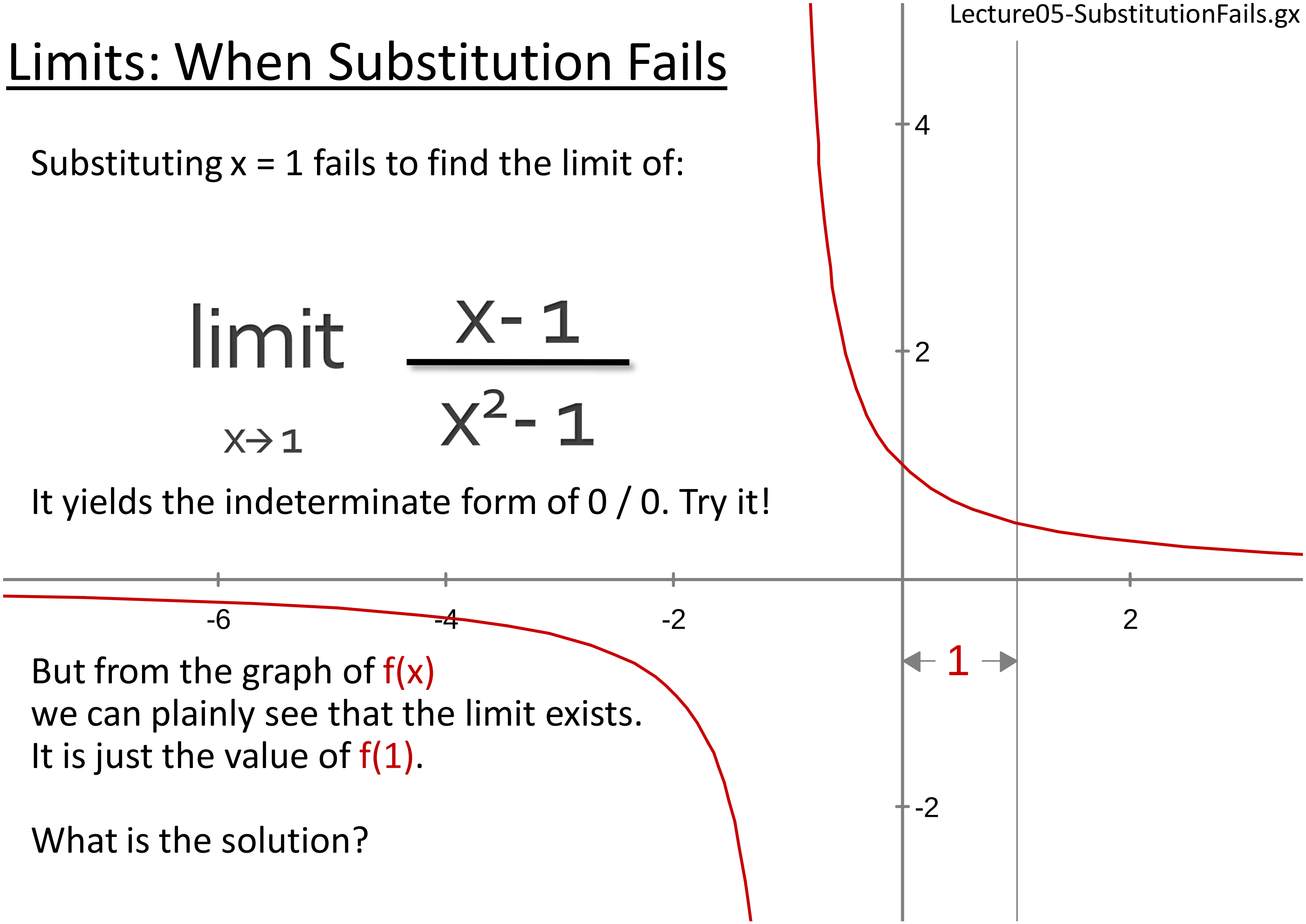


Limits: When Substitution Fails

Substituting x = 1 fails to find the limit of:

$$\lim_{x \to 1} \frac{x - 1}{x^2 - 1}$$

It yields the indeterminate form of 0 / 0. Try it!



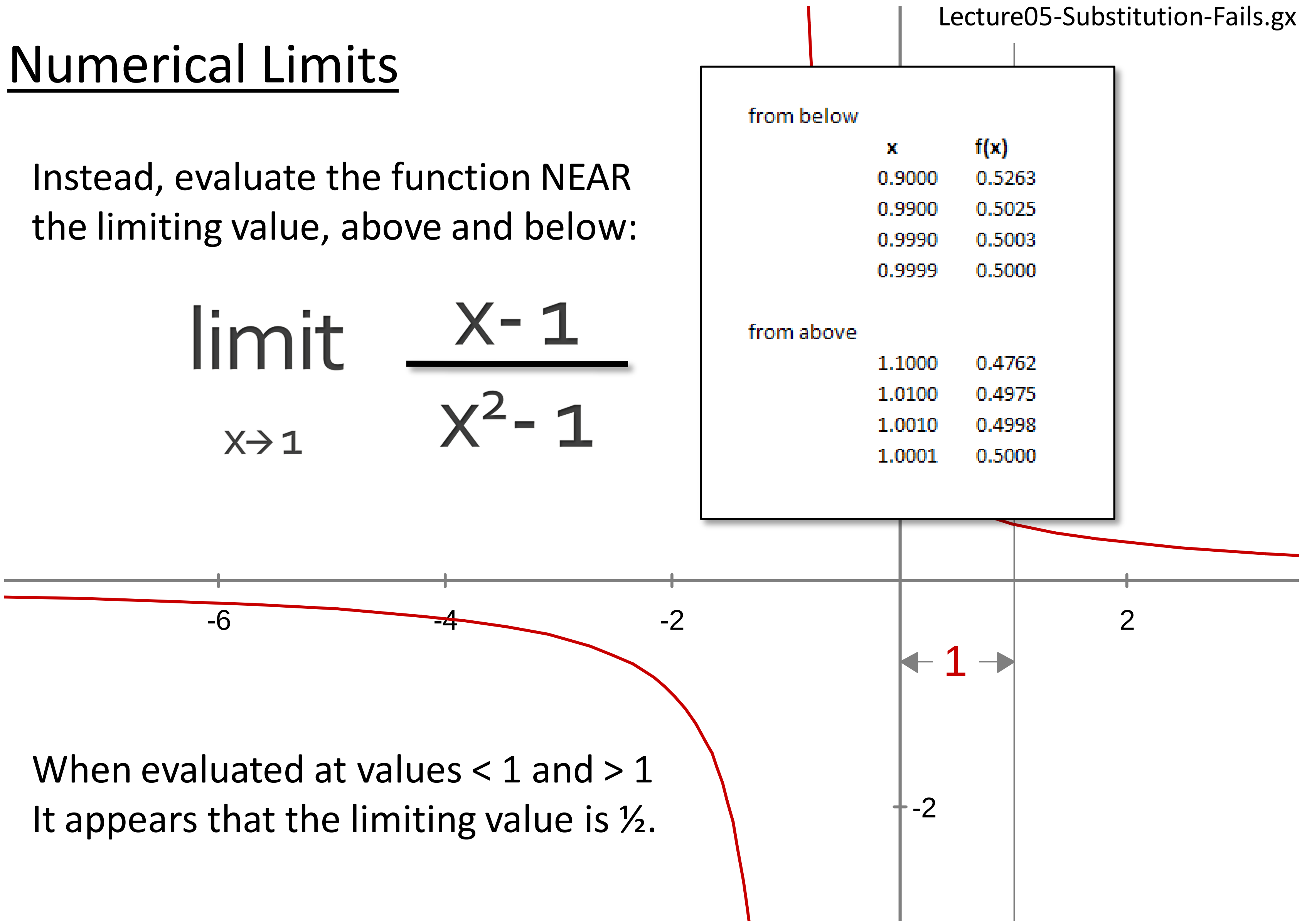
But from the graph of f(x)
we can plainly see that the limit exists.
It is just the value of f(1).

What is the solution?

Numerical Limits

Instead, evaluate the function NEAR the limiting value, above and below:

$$\lim_{x \to 1} \frac{x - 1}{x^2 - 1}$$



from below

x	f(x)
0.9000	0.5263
0.9900	0.5025
0.9990	0.5003
0.9999	0.5000

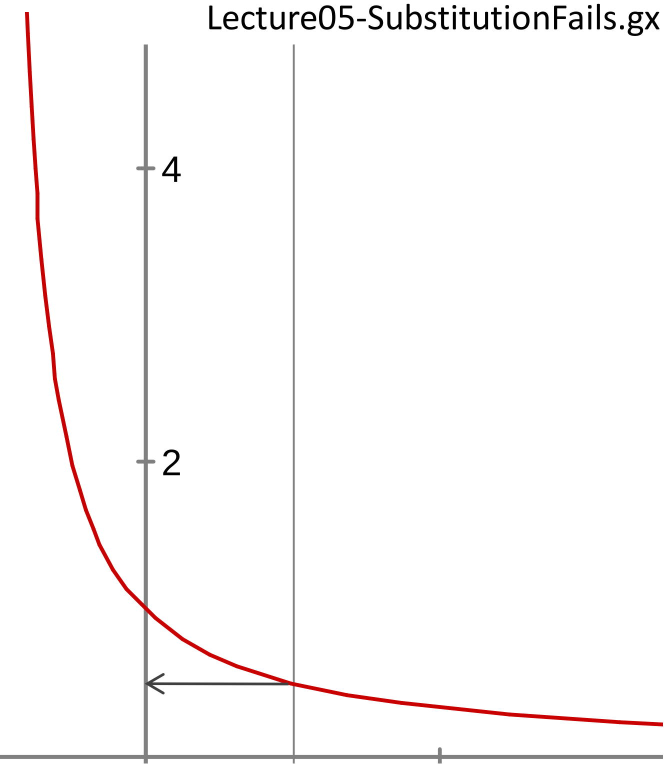
from above

x	f(x)
1.1000	0.4762
1.0100	0.4975
1.0010	0.4998
1.0001	0.5000

When evaluated at values < 1 and > 1
It appears that the limiting value is ½.

Lecture05-SubstitutionFails.gx

Factoring Limits



For functions f(x) with division,
factoring
sometimes works to find the limit.

$$\underset{x \to 1}{\text{Limit}} \frac{x-1}{x^2-1} = \frac{(x-1)}{(x-1)(x+1)} = \frac{1}{(x+1)} = \frac{1}{(1+1)} = \frac{1}{2}$$

Limits of A Function - Laws

Limit of Sum = Sum of Limits

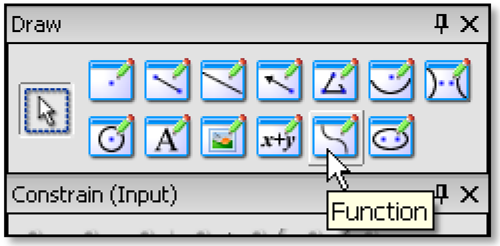
Limit of Difference = Diff. of Limits

Limit of Product = Prod. of Limits*

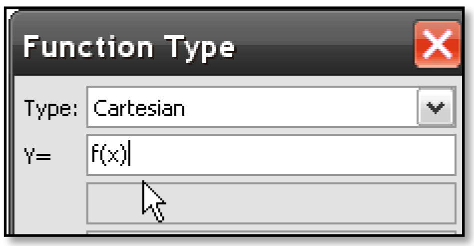
Limit of Quotient = Quo. of Limits

* includes constant case.

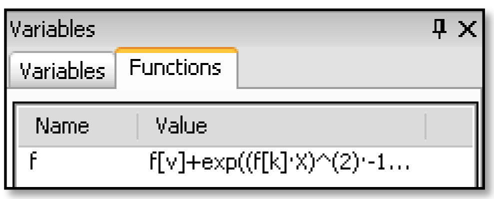
Defining Functions in Geometry Expressions™



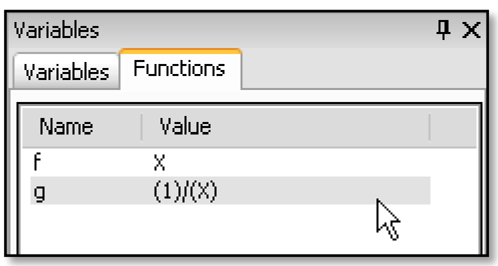
1) Define a function using the Draw→Function button.



2) Use an undefined name like f(x), or g(x).



3) Expressions appear in the Variables→Functions tab.



4) Change the expressions to that desired.

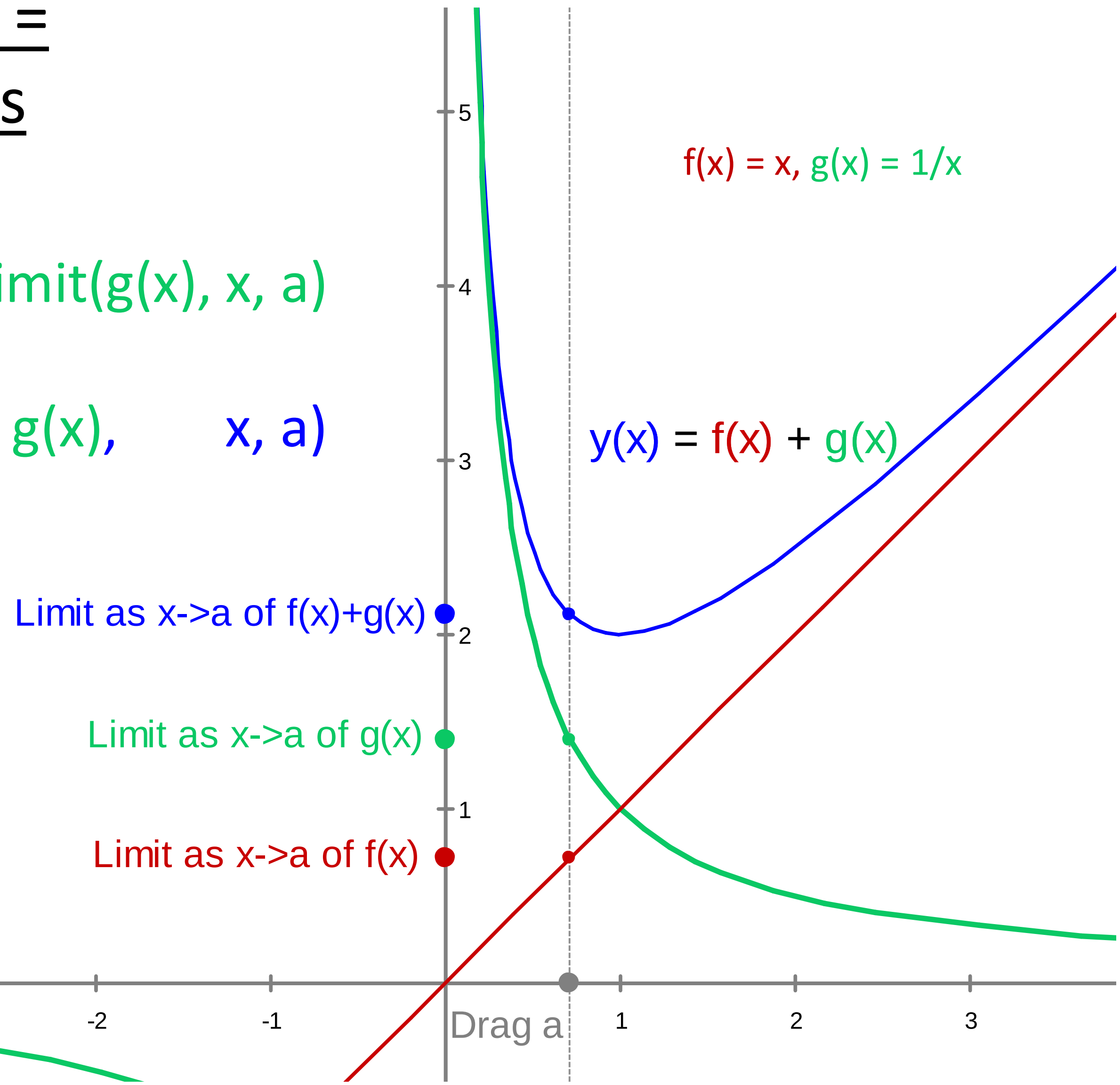
File→Open→Lecture05-LimitLawsSum.gx

Limit of the Sum = Sum of the Limits

Limit(f(x), x, a) + Limit(g(x), x, a)

=

Limit(f(x) + g(x), x, a)



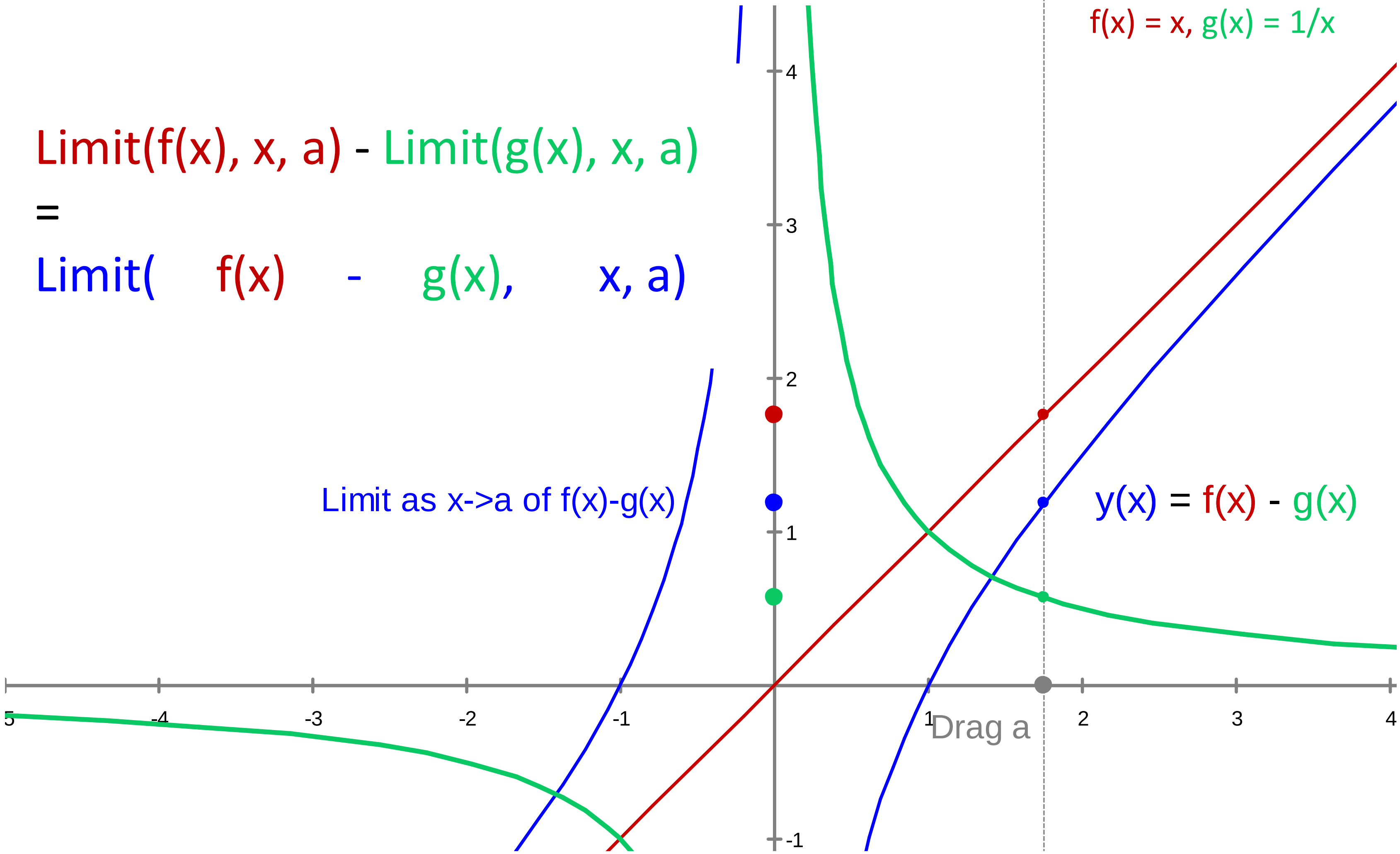
File→Open→ Lecture05-LimitLawsDiff.gx

Limit of the Difference = Difference of the Limits

Limit(f(x), x, a) - Limit(g(x), x, a)

=

Limit(f(x) - g(x), x, a)



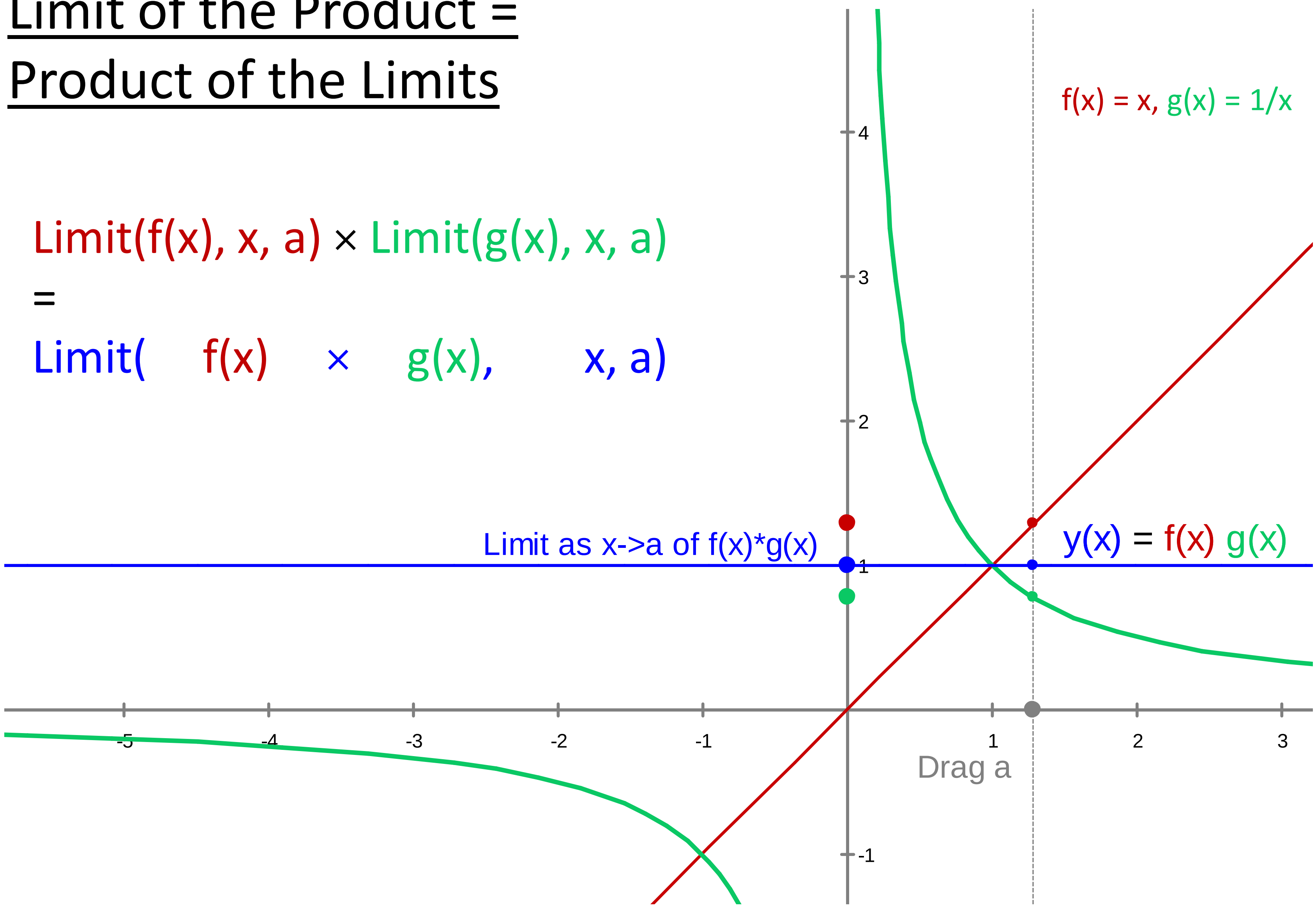
Limit of the Product = Product of the Limits

File→Open→ Lecture05-LimitLawsProd.gx

Limit(f(x), x, a) × Limit(g(x), x, a)

=

Limit(f(x) × g(x), x, a)



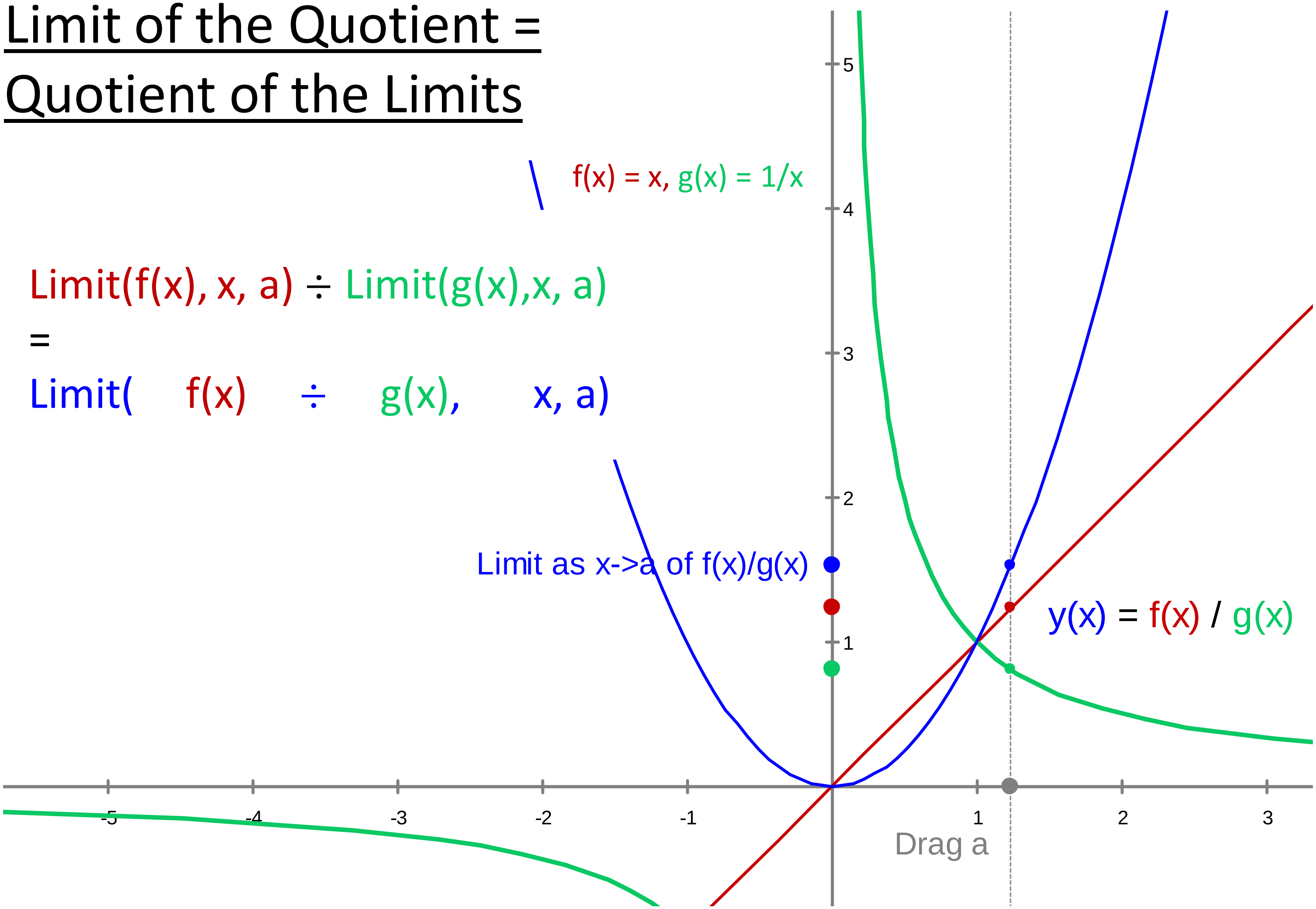
File→Open→ Lecture05-LimitLawsQuo.gx

Limit of the Quotient = Quotient of the Limits

Limit(f(x), x, a) ÷ Limit(g(x),x, a)

=

Limit(f(x) ÷ g(x), x, a)



End