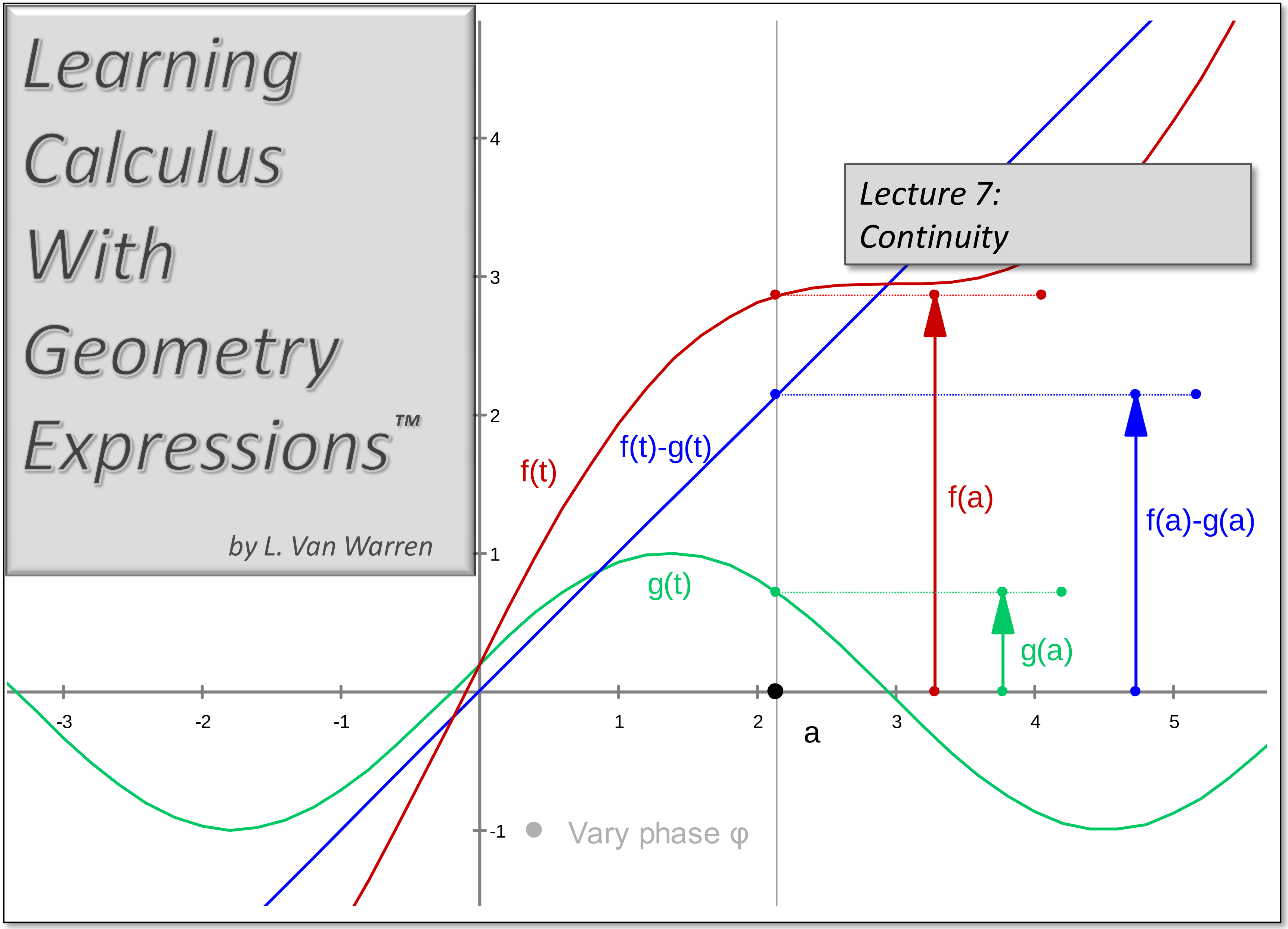
Learning Calculus With Geometry Expressions™

by L. Van Warren

Lecture 7: Continuity



Chapter 2: Limits

Lecture	*Topic*
5	*Finite Limits*
6	*Infinite Limits*
7	***Continuity***
8	*Discontinuity*
9	*Precise Definition of The Limit*

Inspiration

Excerpt from Yutaka Taniyama and His Time:
Very Personal Recollections
Goro Shimura Bull. London Math. Soc. 1989 21 (2), p. 186

Yutaka Taniyama (1927 – 1958)

Yutaka Taniyama was a mathematician at the University of Tokyo. He wore a tailored blue-green suit with a metallic sheen, and neglected to tie his shoelaces.

He made critical observations that would enable the solution of a puzzle over 329 years old – Fermat's last theorem.

Taniyama made a conjecture that linked elliptical algebraic curves with modular, or clock face arithmetic.

Proof of his conjecture would come 39 years after he made it and would enable Andrew Wiles, with help from his student Richard Taylor to finally solve Fermat's Last Theorem.

Taniyama was lost tragically, but the effect of his genius would influence generations.

Continuity

As we use tools like finite and infinite limits, certain conditions must be set on the functions being analyzed. Since we cannot be at infinity to examine the behavior of a function, we must insure these conditions are met as we travel towards it!

One intuitive property of a curve is its continuity. Is it smooth, or does it jump, break or peak sharply?

The idea and definition of continuity applies to the original curve – the ***position*** of y values for certain x values (C_0 continuity). C_1 continuity considers how fast the ***slope*** of a curve changes. The curve of slopes is a new curve in its own right and we will study it closely later. This idea is extensible to include the ***acceleration*** of y values with respect to x values, and this becomes C_2 continuity. It is said that the eye can see variations in shape up to C_4.

Continuity Definition

A function f(x) is continuous at **a** if:

$$\lim_{x \to a} f(x) = f(a)$$

This is a simple but powerful definition.

There are three conditions that must be satisfied:

1) f(a) is defined and exists
2) Limit(f(x), x → a) exists
3) Limit(f(x), x → a) = f(a)

Laws of Continuity – Constant Case

Now is a good time to see different kinds of notation you will encounter in mathematical and computer science literature. We will write the same idea three different ways, starting with the most verbose and ending with the most compact.

Statement 1:

If the function ***f(x)*** is continuous then the function ***c ·f(x)*** is also continuous.

Statement 2:

The function ***isContinuous(f(x))*** returning true implies ***isContinuous (c· f(x))*** will also.

Statement 3:

isContinuous(f) → ***isContinuous (c· f)***

In the last statement, ***f*** being a function of ***x*** is a given and understood. It has the same meaning as the previous two statements.

Lecture07-ContinuityOfConstant.gx

Laws of Continuity – Constant Case

Here is an example of the law of continuity for a function ***f(x)*** times a constant ***c***.

EXERCISES

1) Change the value of ***c*** in the *Variables* dialog box from 2 to 3. How does the curve change?

Variables — Variables | Functions

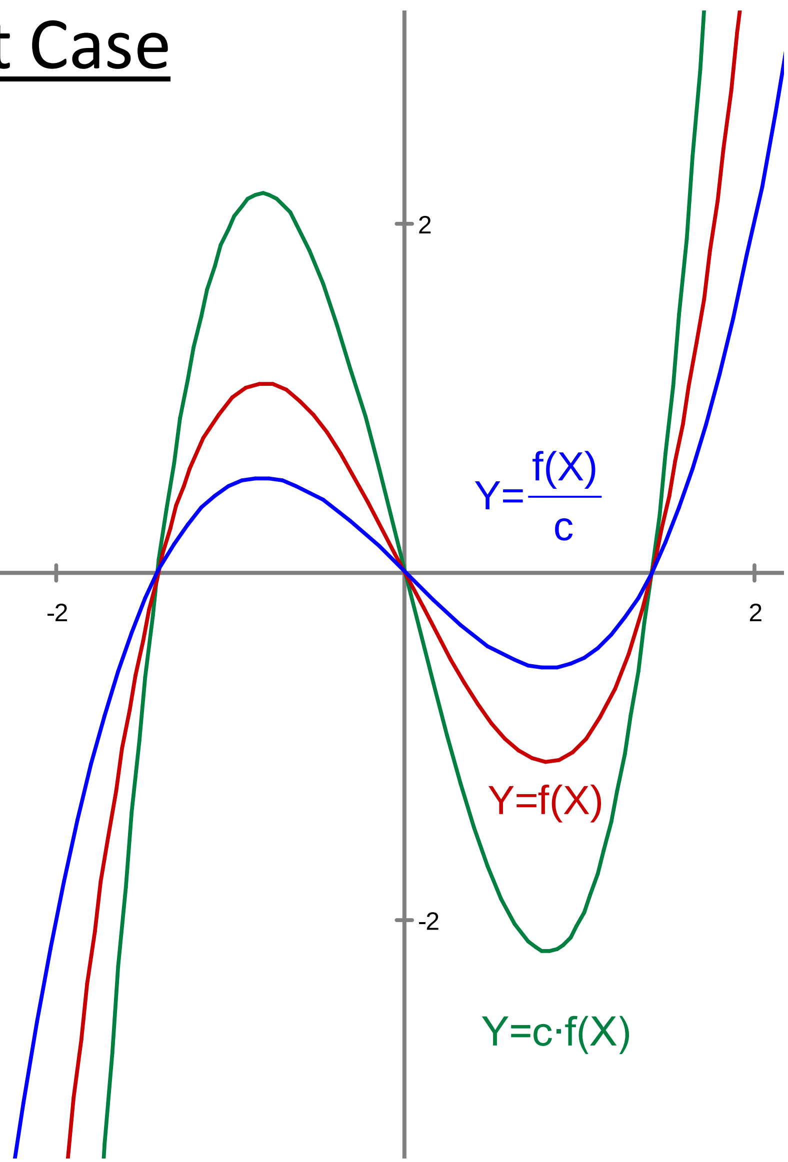
Name	Value	Locked
c	2	-

2) Change the definition of f(x) in the *Functions* dialog box.

3) Does the Continuity Law remain true for your example?

Variables — Variables | Functions

Name	Value
f	-2·X+X^(3)



Laws of Continuity For Function Pairs

If the pair of functions ***f(x)*** and ***g(x)*** are continuous, their sum, difference, product and quotient are also continuous. As before consider three different ways of stating this. The last statement is the most general.

Statement 1:

If ***f(x)*** and ***g(x)*** are continuous then the function ***h(x)*** = ***f(x)*** + ***g(x)*** is also continuous.

Statement 2:

isContinuous(f(x))) AND isContinuous(g(x)) implies isContinuous (f(x) + g(x))

Statement 3:

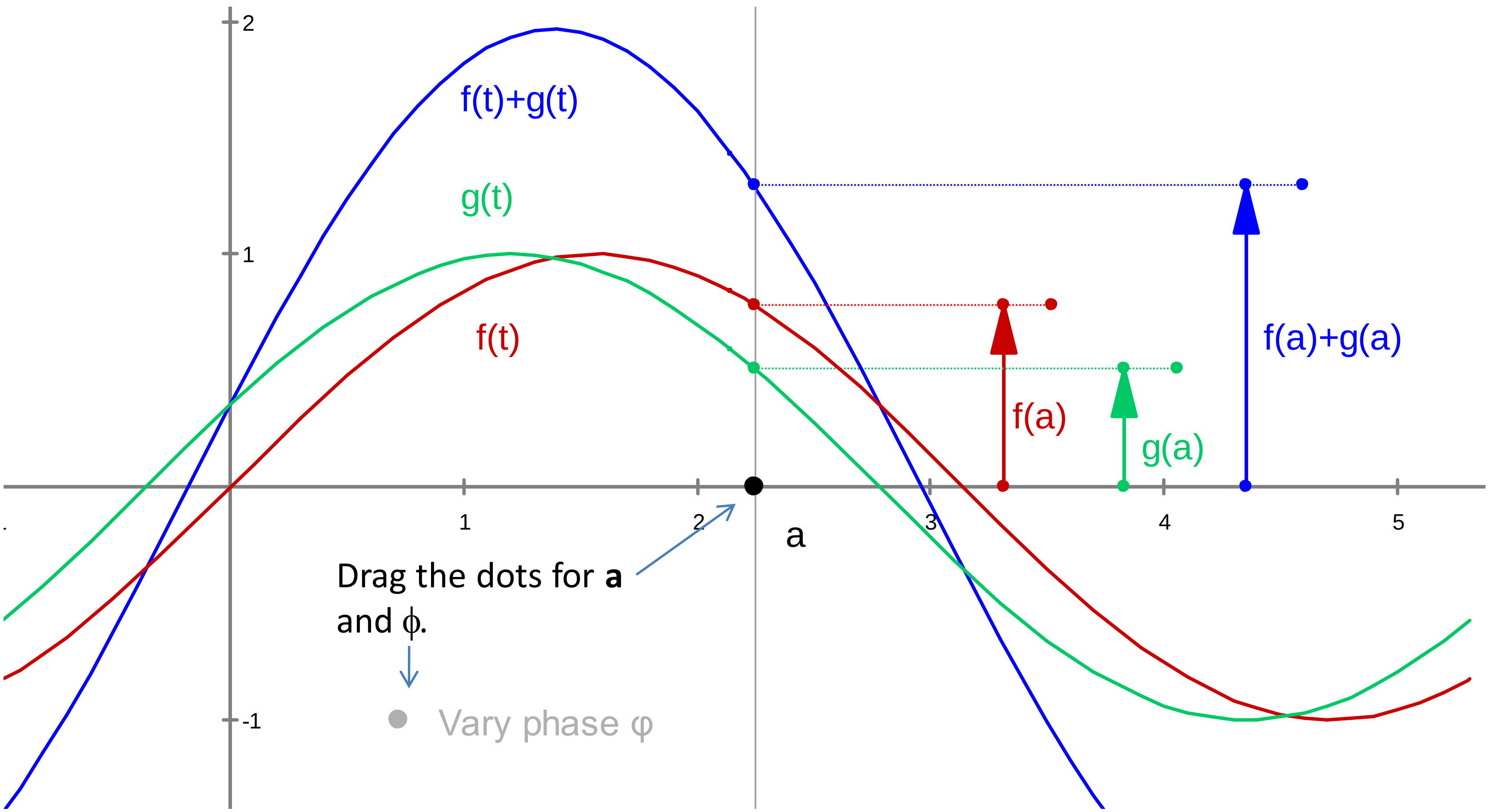
isContinuous(f(x))) ∩ isContinuous(g(x)) → isContinuous (f(x) ⊗ g(x))
where ⊗ = { +, -, •, ÷}

In the last statement we are using the ⊗ symbol to stand for addition, subtraction, multiplication and division.

Lecture07-ContinuityOFSum.gx

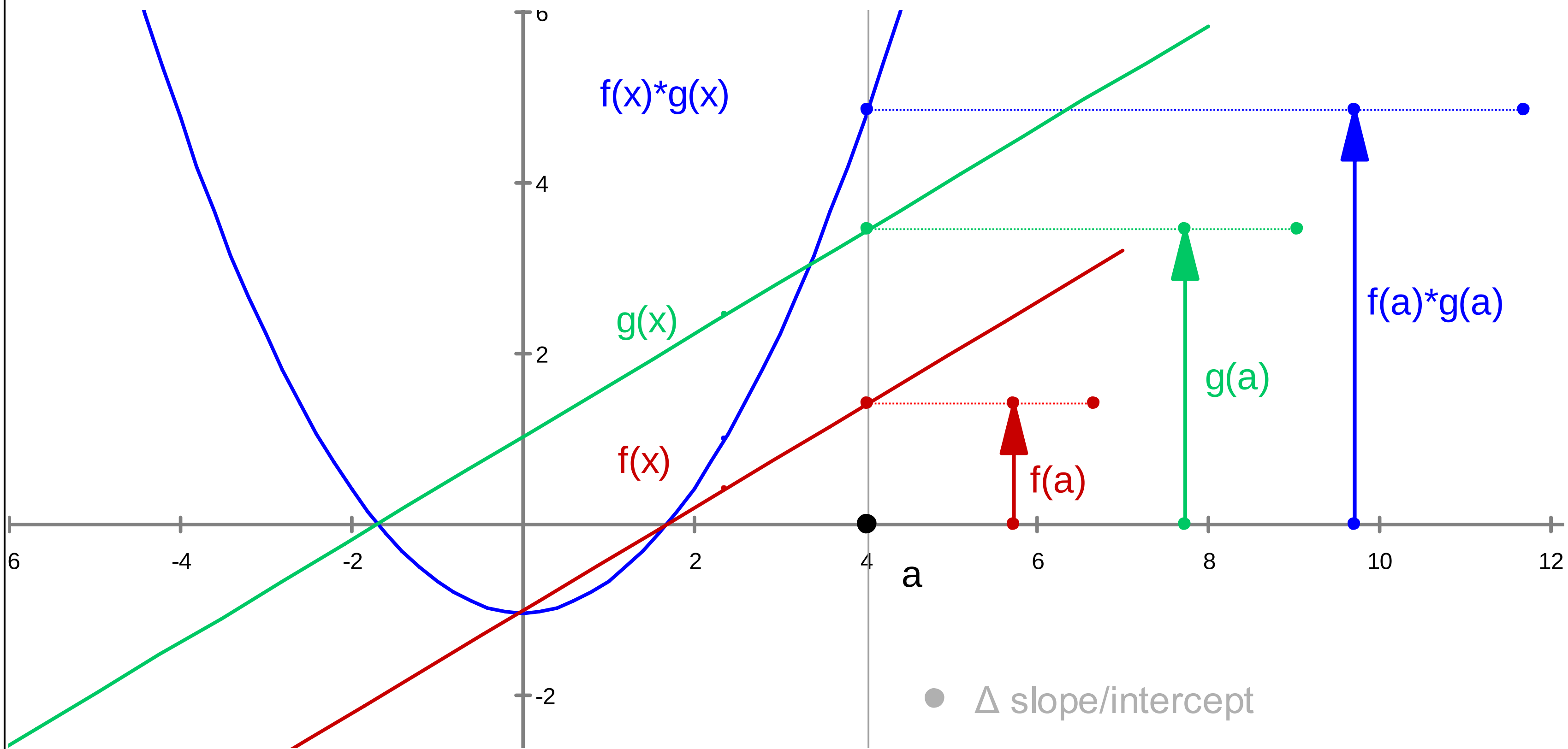
Laws of Continuity for Summed Pair

This example shows the Law of Continuity for the sum of two sinusoidal functions, ***f(t)*** and ***g(t)***, added to form a new periodic function ***f(t)* + *g(t)***, also continuous at ***a***.



Laws of Continuity for Product Pair

This example shows the Product Law of Continuity for a pair of linear functions, ***f(x)*** and ***g(x)***, multiplied together to form a quadratic function ***f(x) · g(x)***.



Laws of Continuity for Product Pair

The example on the preceding page showed the Product Law of Continuity acting for a pair of linear functions, ***f(x)*** and ***g(x)***, multiplied together to form a quadratic function ***f(x)*** · ***g(x)***.

Dragging the point labeled ***a*** along the x-axis changes the value at which the limit is being taken. All three functions may be evaluated by taking the limit as x approaches ***a*** from either direction.

Dragging the point labeled ***Δ Slope/Intercept*** changes the slope and y-intercept of the functions. Notice that the limit always exists and the three functions remain continuous for dramatic changes in the shapes of the curves.

Lecture07-ContinuityDifference.gx

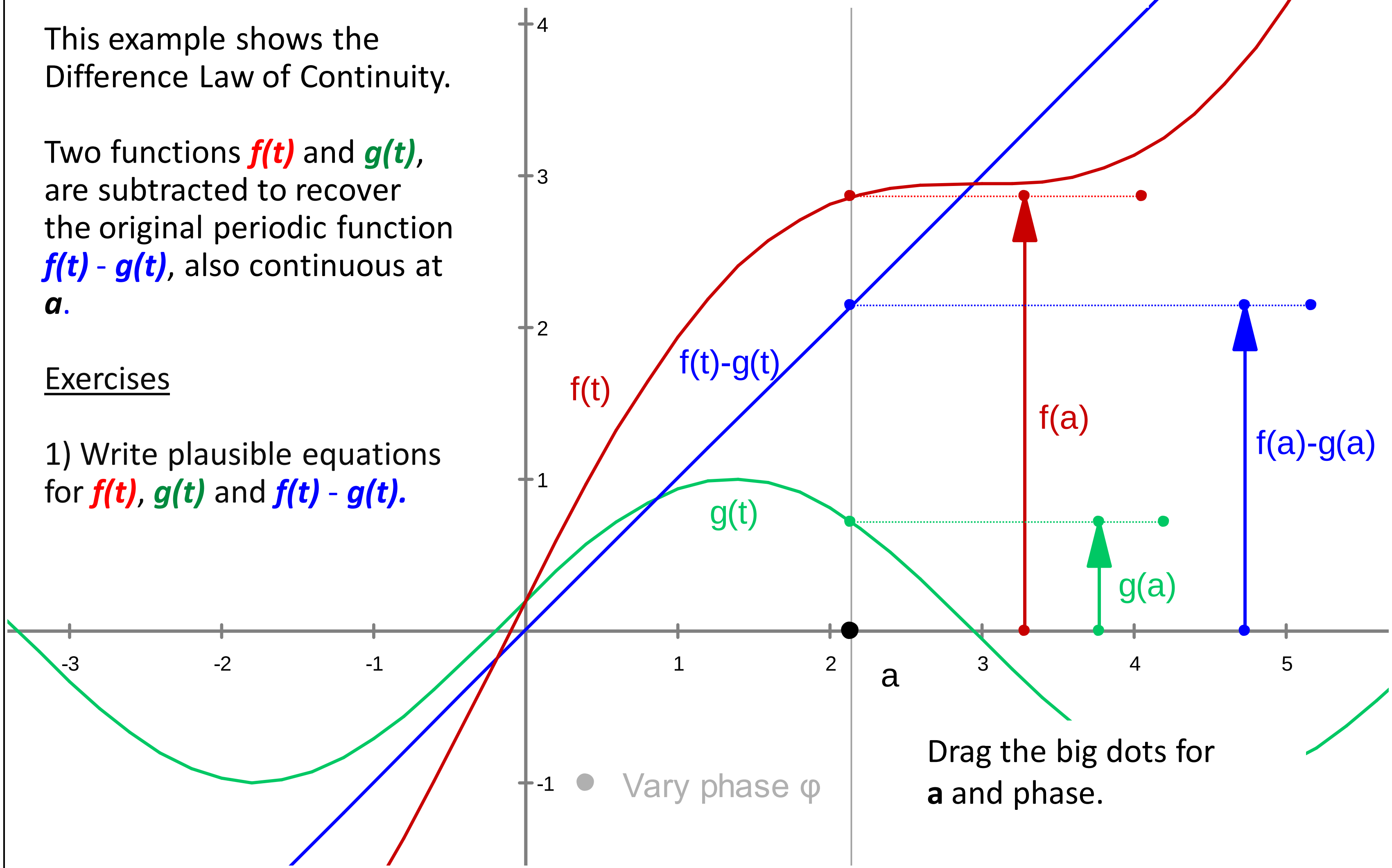
Laws of Continuity for Differenced Pair

This example shows the Difference Law of Continuity.

Two functions ***f(t)*** and ***g(t)***, are subtracted to recover the original periodic function ***f(t)*** - ***g(t)***, also continuous at ***a***.

Exercises

1) Write plausible equations for ***f(t)***, ***g(t)*** and ***f(t)*** - ***g(t)***.



Drag the big dots for **a** and phase.

Laws of Continuity for Divided Pair

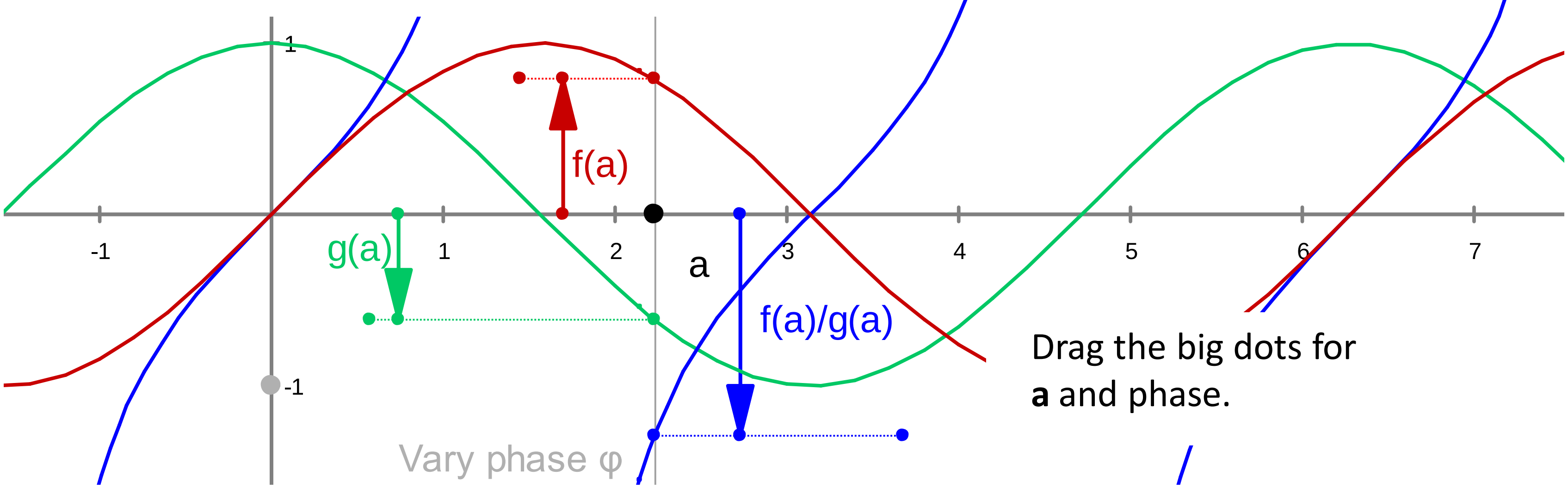
This example shows the Quotient Law of Continuity for two periodic functions, ***f(t)*** and ***g(t)***, divided to create a new periodic function ***f(t)/g(t)***, indeed continuous at ***a***, **but not continuous everywhere.**

EXERCISE
Compute the ***x*** values at which the quotient function is not continuous. Hint: Trigonometry says that for this specific case: ***f(t)/g(t) = tan(t).***

$t,\ f(t) = (t, \sin(t+\varphi))$

$t,\ g(t) = (t, \cos(t+\varphi))$

$t,\ f(t)/g(t) = \left[t, \dfrac{\sin(t+\varphi)}{\cos(t+\varphi)}\right]$



One-Sided Continuity

Up till now we have considered functions that are continuous as we approach some value of ***x* = *a***, from either direction. This is from the definition of continuity which was evaluated by taking the ***limit(f(x), x→a)*** with ***x*** approaching ***a***, from the left or the right. If this limit was equivalent to ***f(a)*** we say the function is continuous at ***x* = *a***.

But we could break this seemingly atomic idea down further into two cases. In one we ask if the limit exists as we approach from the left, from a value of ***x*** less than the value of ***a***. We would write this definition in traditional notation as:

$$\lim_{x \to a^-} f(x) = f(a)$$

where the a^- implies that ***x*** approaches from the left. We could also write ***limit(f(x), x→a⁻)*** or as ***limit(f(x), x, a⁻)*** understanding that these statements refer to the same underlying idea.

EXERCISE
State the remaining case where we take the limit by approaching ***a*** from the right rather than the left. Use the notation ***limit(f(x), x→a⁺)*** to state this definition.

Two-Sided Continuity From One-Sided Continuity

The two-sided definition of continuity comes by joining the two one-sided pieces with a Boolean combination, a conjunction of the two original statements, using AND:

{ limit(f(x), x→a) = f(a) } implies {limit(f(x), x→a^-) =f(a) AND limit(f(x), x→ a^+)=f(a)}

In pure functional notation we replace change limit notation from arrows to arguments:

isEquivalent(limit(f(x), x, a) = f(a), AND(limit(f(x), x, a^-) , limit(f(x), x, a^+))

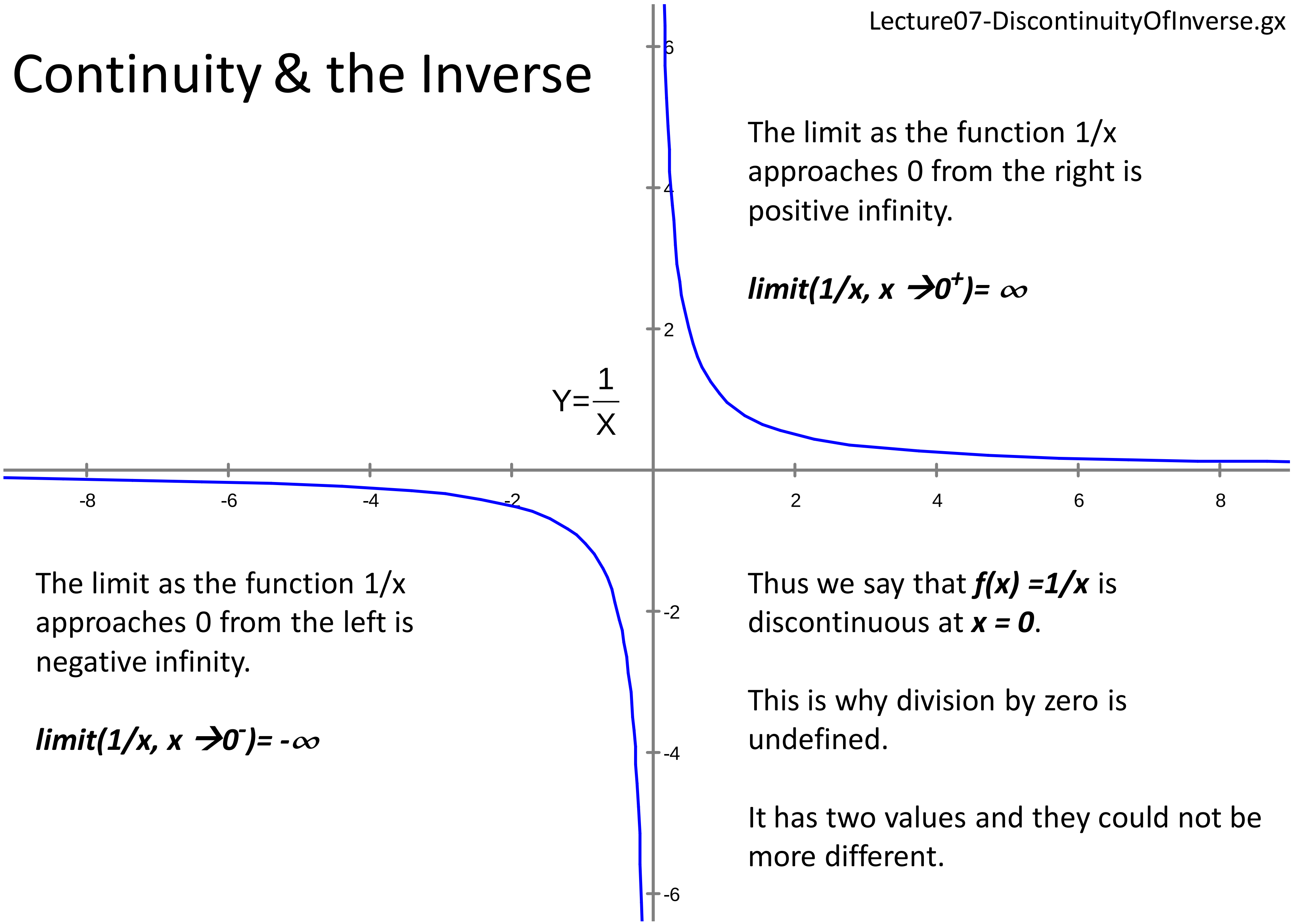
This says,
"If the answer you get from the left is the same as the one you get from the right then the function is continuous. "

EXERCISES

1) Write the converse of the first statement using the same notation.
2) Construct a scenario which does not satisfy this and view the discontinuity.

Lecture07-DiscontinuityOfInverse.gx

Continuity & the Inverse



The limit as the function 1/x approaches 0 from the right is positive infinity.

limit(1/x, x →0$^+$)= ∞

The limit as the function 1/x approaches 0 from the left is negative infinity.

limit(1/x, x →0$^-$)= -∞

Thus we say that ***f(x) =1/x*** is discontinuous at ***x = 0***.

This is why division by zero is undefined.

It has two values and they could not be more different.

Problem Solving Strategies - Divide and Conquer

Being able to break down the definition of continuity into two cases is an example of the problem solving strategy of *divide and conquer*. The intervals are ***(-∞, a]*** and ***[a, ∞)*** .

Even though we examine each case sequentially and therefore consecutively, there is nothing demanding that we prioritize one before the other. The two cases could be handled simultaneously, i.e. *concurrently*, if we had the ability to process two ideas with equal priority at the same time.

When we break out the two cases, an ***artificial sequentiality*** is produced by the fact that we process tasks one at a time. It is important to recognize that the sequential nature of processing is an artifact of the process we use to solve the problem, and is not intrinsic to the problem itself. Artificial sequentiality can introduce bias into our problem solving, causing some cases to receive more attention than they deserve because of the order in which they were examined. I call this informally, "***the curse of sequentiality***". This along with the fact that we remember what we last did better than what we first did can limit our problem solving abilities.

Trick: Bringing Terms Out of Degeneration

Another problem-solving strategy, or trick that exploits a situation is that of bringing terms out of degeneration. We use this all the time, often without recognizing it.

For example when we rationalize the denominator of a fraction:

$$\frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

We are multiplying the target by different versions of the number 1. Its just that 1 is represented by successively more complex terms that accomplish some overall objective. So we say that the number 1 is "coming out of degeneration".

One of my favorite examples, that we will be using soon is that of recognizing that:

$$x^2 - 1 = (x+1)\cdot(x-1)$$

In this case the first expression "comes out of degeneration" to create a form that enables simplification in the long run.

Polynomial Continuity

From the limit laws, for example, the limit of the sum is the sum of the limits, and continuity laws it can be shown that a polynomial of the form:

$$y = a + bx + cx^2 + \ldots$$

is continuous everywhere.

The rational polynomial:

$$y = \frac{a_0 + a_1 x + a_2 x^2 + \ldots}{b_0 + b_1 x + b_2 x^2 + \ldots}$$

is continuous everywhere that the denominator is non-zero. We can write rational polynomials in a variety of ways to suit our needs as the next example shows.

EXERCISE
Prove the law of polynomial continuity. Use the web. Search is your friend.

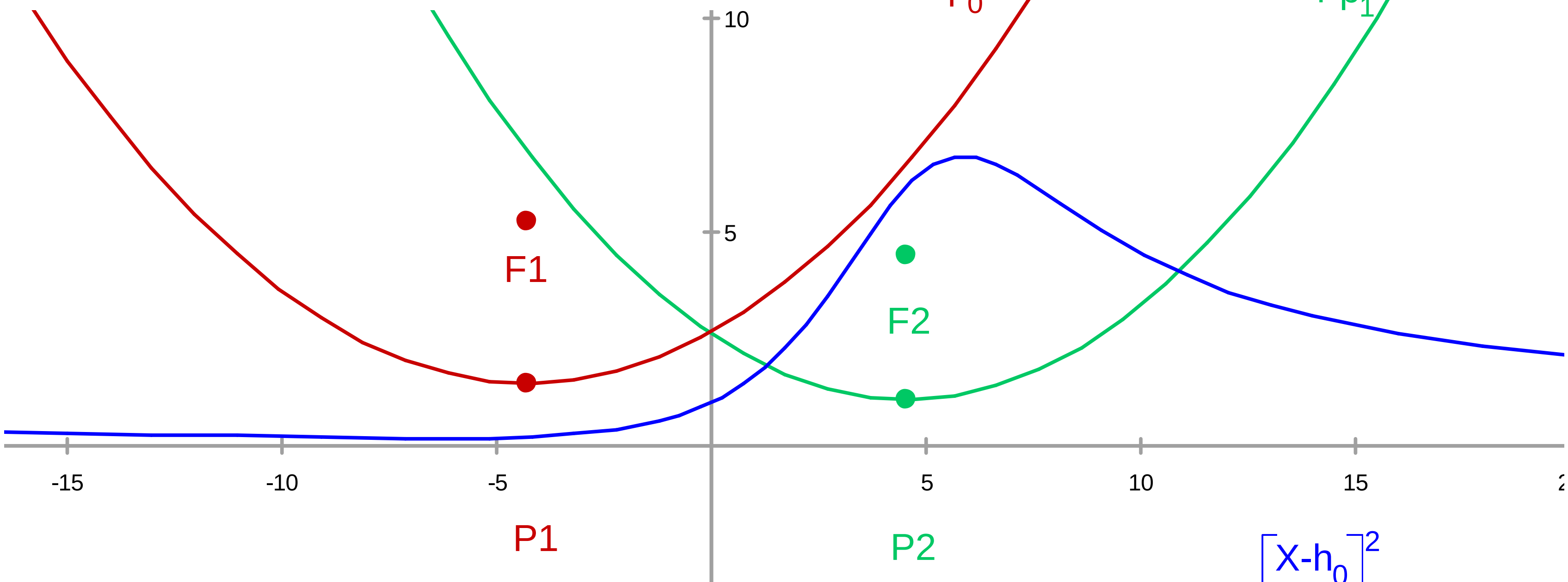
Lecture07-RationalPolynomial.gx

Rational Polynomials

A rational polynomial is the ratio of two polynomials.

$$Y = k_0 + \frac{[X-h_0]^2}{4 \cdot p_0}$$

$$Y = k_1 + \frac{[X-h_1]^2}{4 \cdot p_1}$$

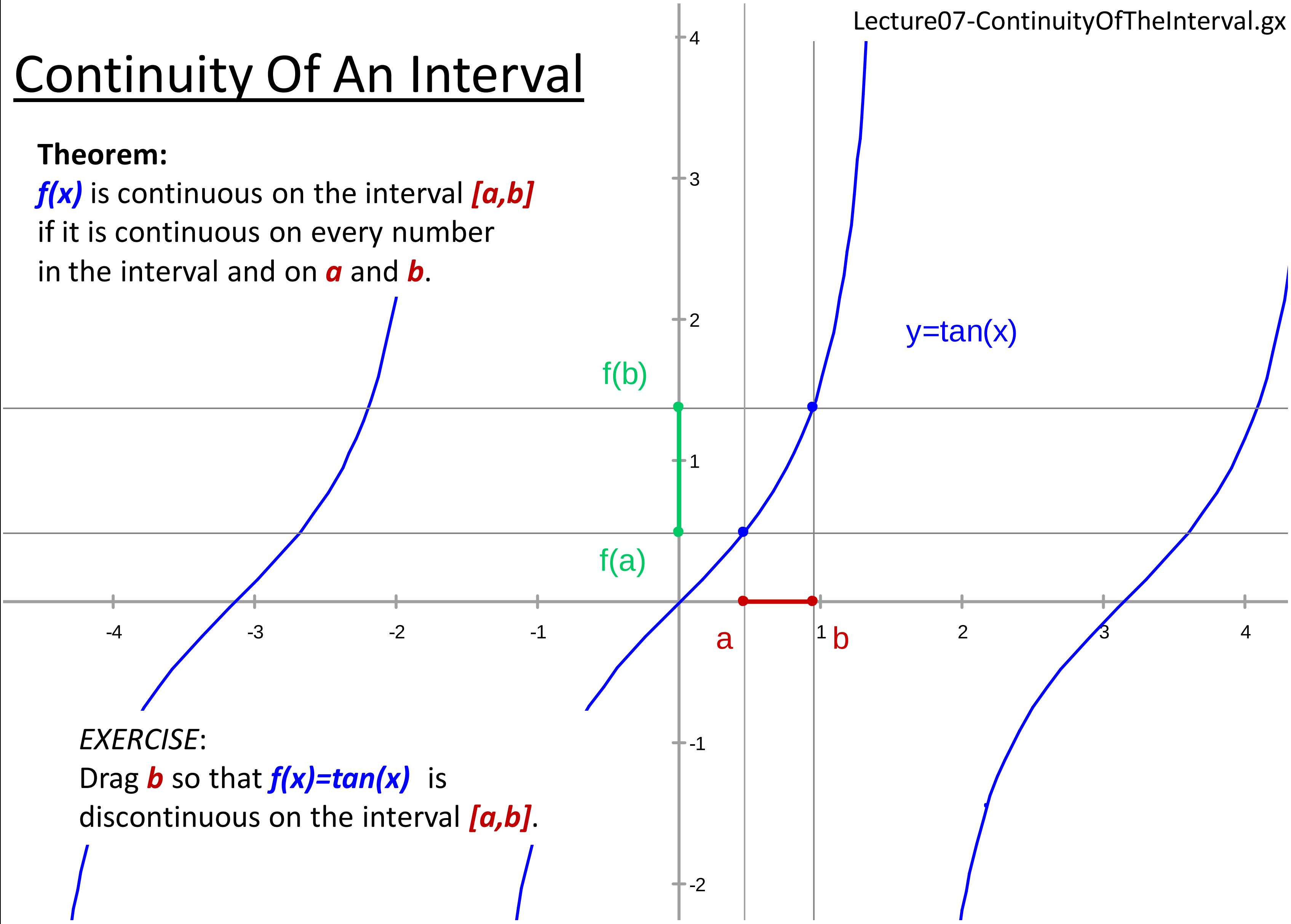


EXERCISE:
Where is the rational polynomial discontinuous?

$$Y = \frac{k_0 + \frac{[X-h_0]^2}{4 \cdot p_0}}{k_1 + \frac{[X-h_1]^2}{4 \cdot p_1}}$$

Continuity Of An Interval

Theorem:
f(x) is continuous on the interval ***[a,b]***
if it is continuous on every number
in the interval and on ***a*** and ***b***.



EXERCISE:
Drag ***b*** so that ***f(x)=tan(x)*** is
discontinuous on the interval ***[a,b]***.

Lecture07-SqueezeTheorem1.gx

The Squeeze Theorem:

Sometimes we have to find the limit of a function ***g(x)***, whose behavior is more complex than that of two simpler bounding functions, ***f(x)*** and ***h(x)***.

In this situation we can invoke the Squeeze Theorem which states:

For ***x*** in an interval near ***a,***

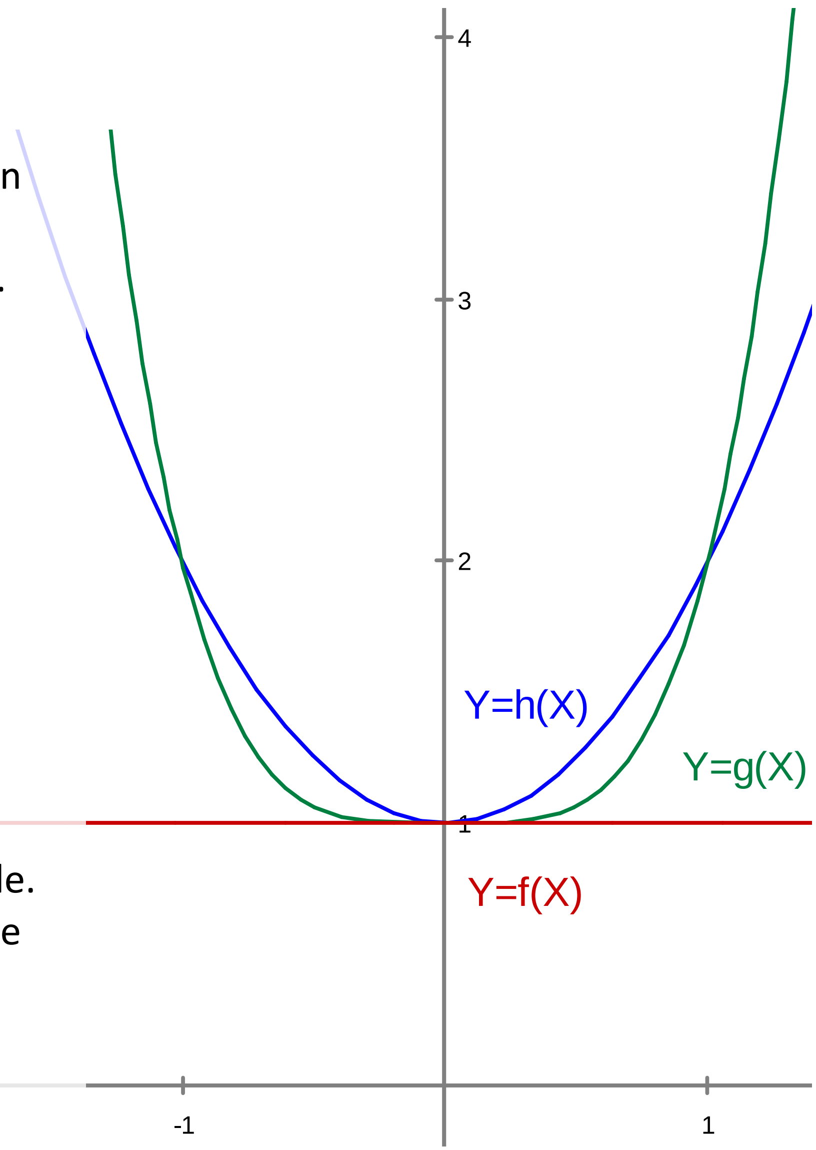
if ***f(x) ≤ g(x) ≤ h(x)***

and

lim (f(x) ,x→a) = lim (h(x),x→a)

then

lim (g(x),x→a)



In this example ***a=0*** and ***g(x)*** was relatively simple. Observe that who squeezes who depends on the interval under consideration as ***h(x)*** and ***g(x)*** change places outside the interval [-1, 1].

The Squeeze Theorem:

Consider the more dramatic example:

$f(x) = 10\,x^2$

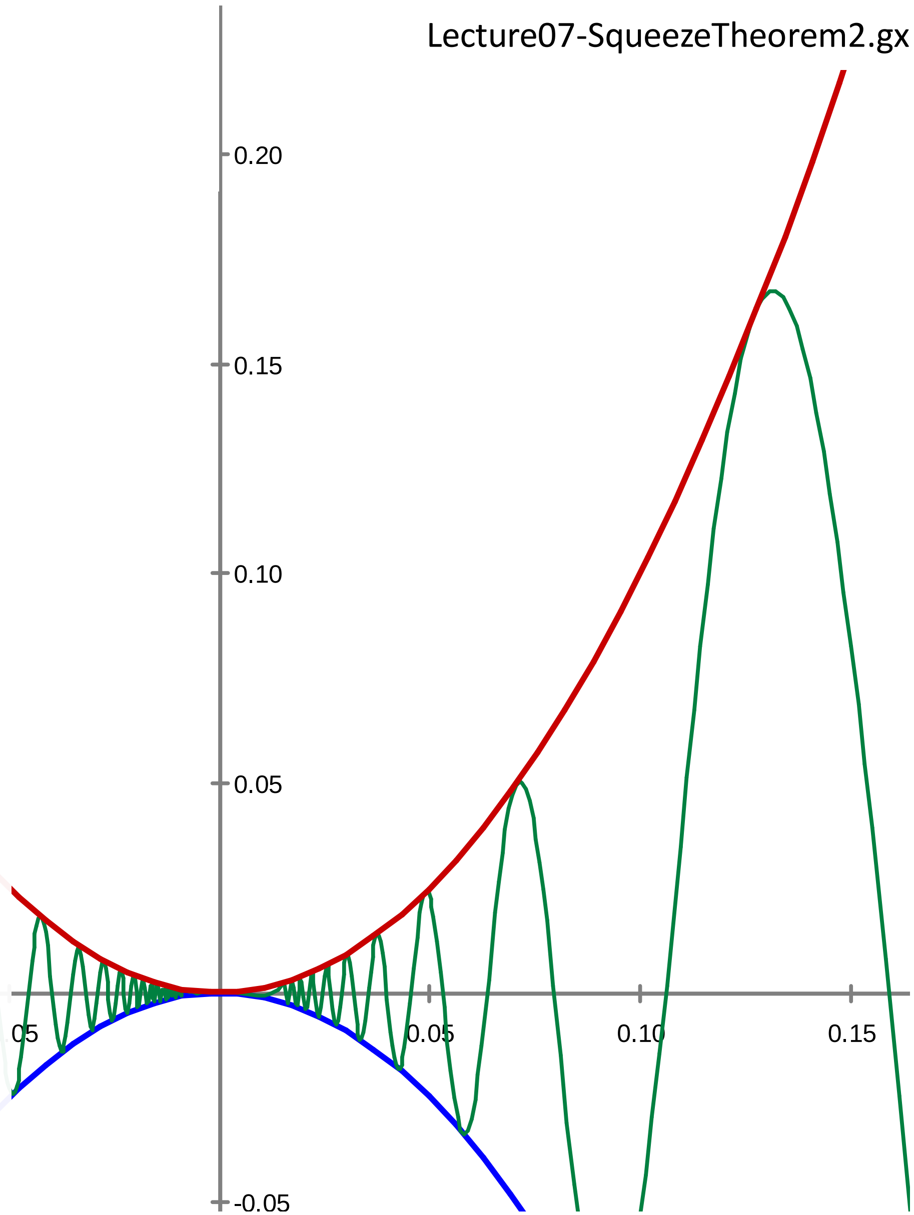
$h(x) = -10\,x^2$

Squeezing : $g(x) = 10\,x^2 \cdot \sin(1/x)$

Observe that we have to zoom way in to see the true behavior of the function near the origin. For this case ***x=a=0***.

EXERCISE

Rewrite the equations to demonstrate the same argument at ***x=a=1***.



Continuity In Composition of Functions

Given a function f(x) and g(x) we can compose the two to make a new function

$$h(x) = f(g(x)) \qquad OR \qquad h(x) = g(f(x))$$

If as we approach x = a, the limit of f(x) and g(x) exists, **then** the limit of f(g(x)) exists also.

We can write the first composition in mumbo jumbo as:

IF (exists(limit(f(x), x, a)) AND exists(limit(g(x), x, a)) THEN exists(limit(f(g(x)), x, a))

Or even more compactly with an implied IF:

exists(limit(f(x), x, a)) ∩ exists(limit(g(x), x, a)) → exists(limit(f(g(x)), x, a))

EXERCISE

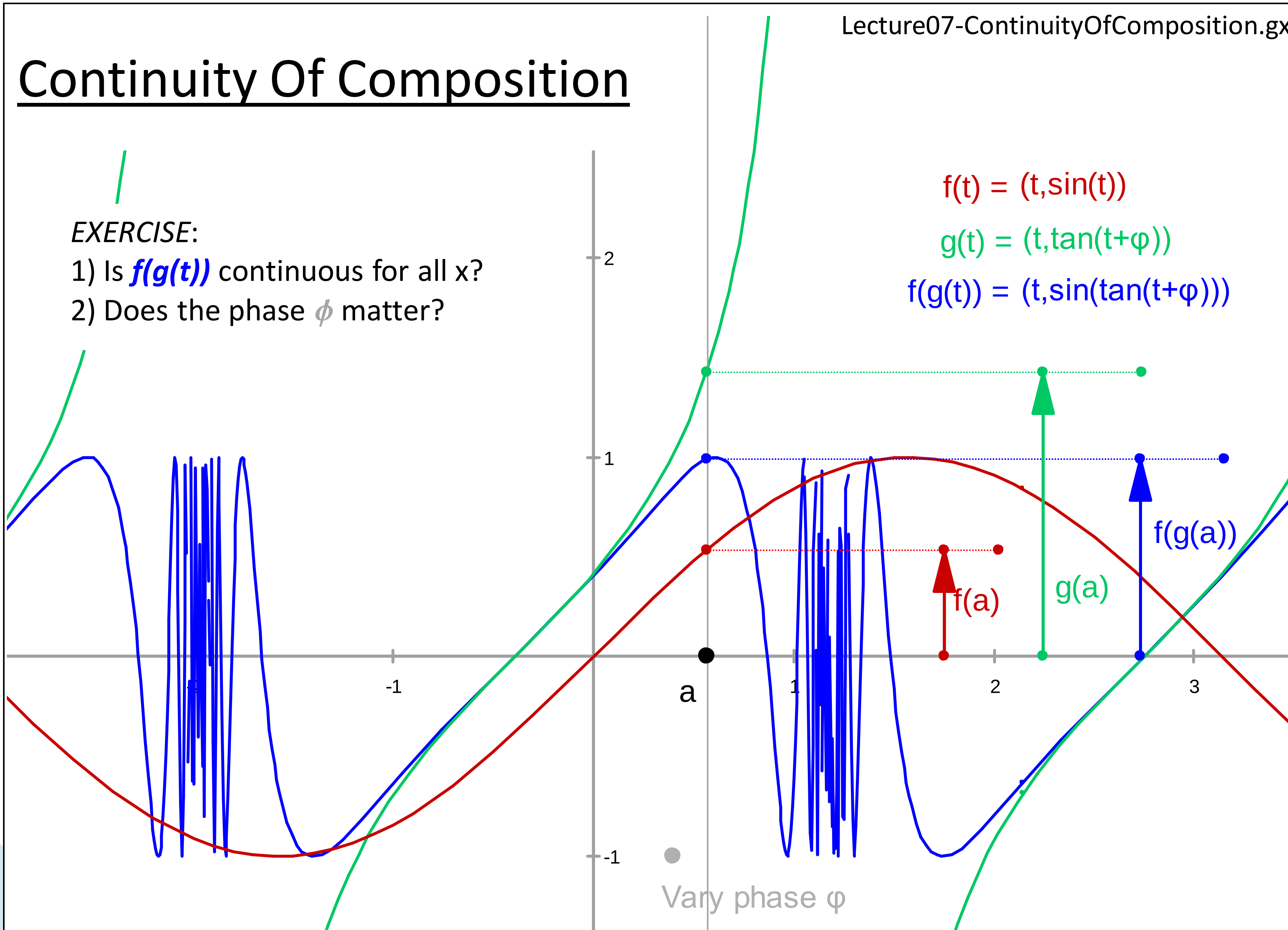
1) Is the converse always true?
2) If not, give an example where it is false.
3) Write the second composition in mumbo jumbo.

Continuity Of Composition

EXERCISE:

1) Is ***f(g(t))*** continuous for all x?

2) Does the phase ϕ matter?

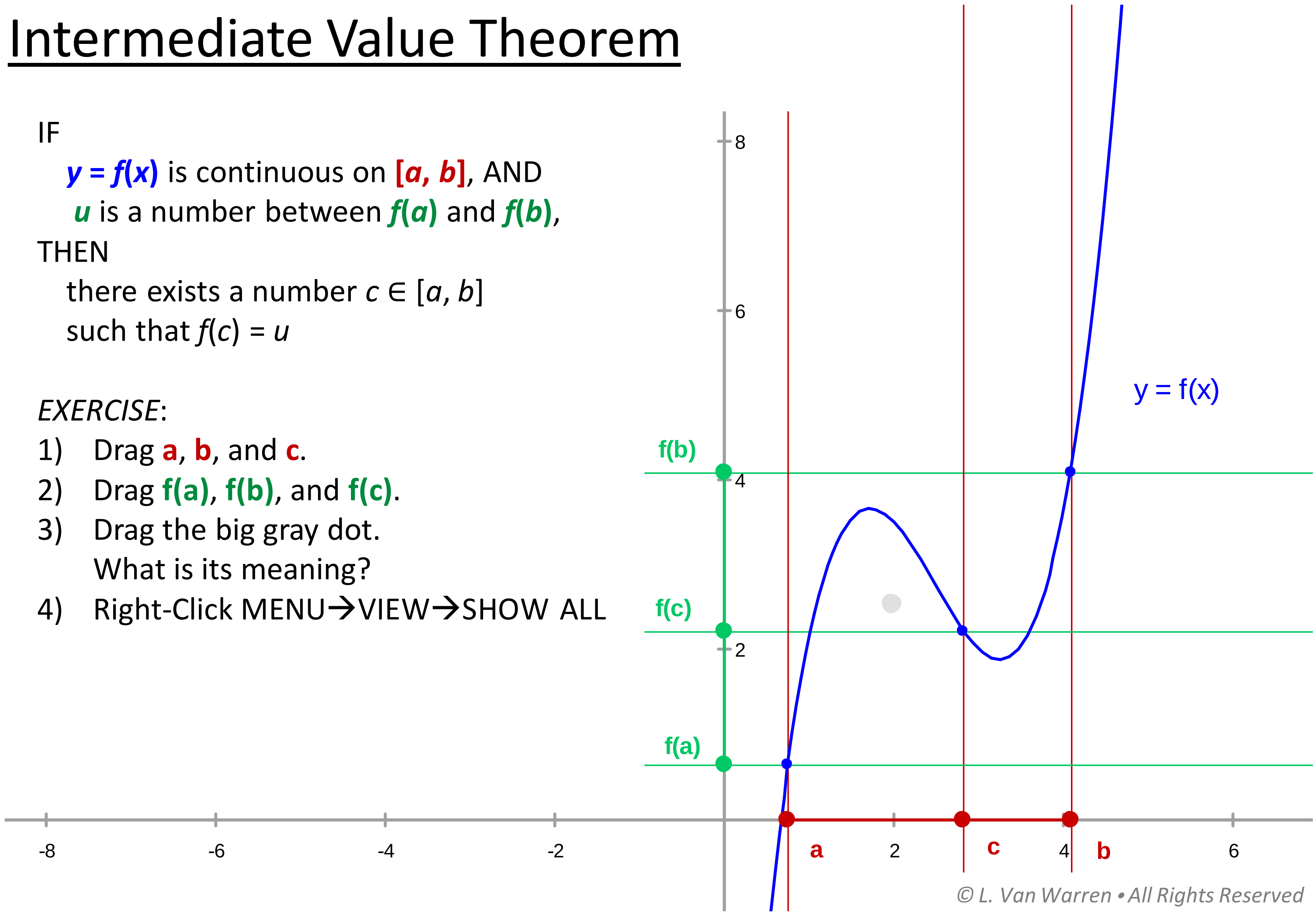


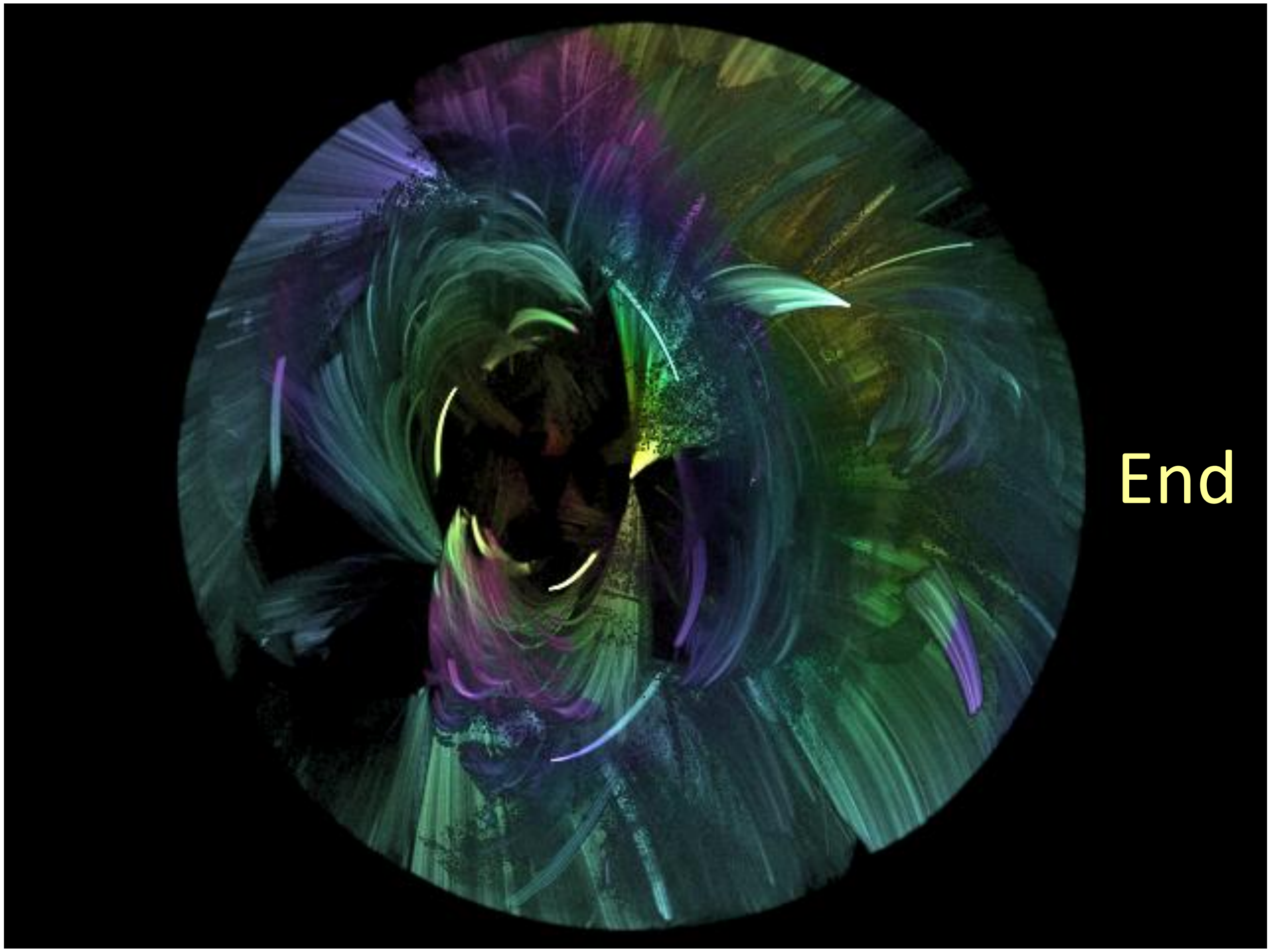
Lecture07-IntermediateValueThereom.gx

Intermediate Value Theorem

IF
$y = f(x)$ is continuous on $[a, b]$, AND
u is a number between $f(a)$ and $f(b)$,
THEN
there exists a number $c \in [a, b]$
such that $f(c) = u$

EXERCISE:
1) Drag **a**, **b**, and **c**.
2) Drag **f(a)**, **f(b)**, and **f(c)**.
3) Drag the big gray dot. What is its meaning?
4) Right-Click MENU→VIEW→SHOW ALL




End