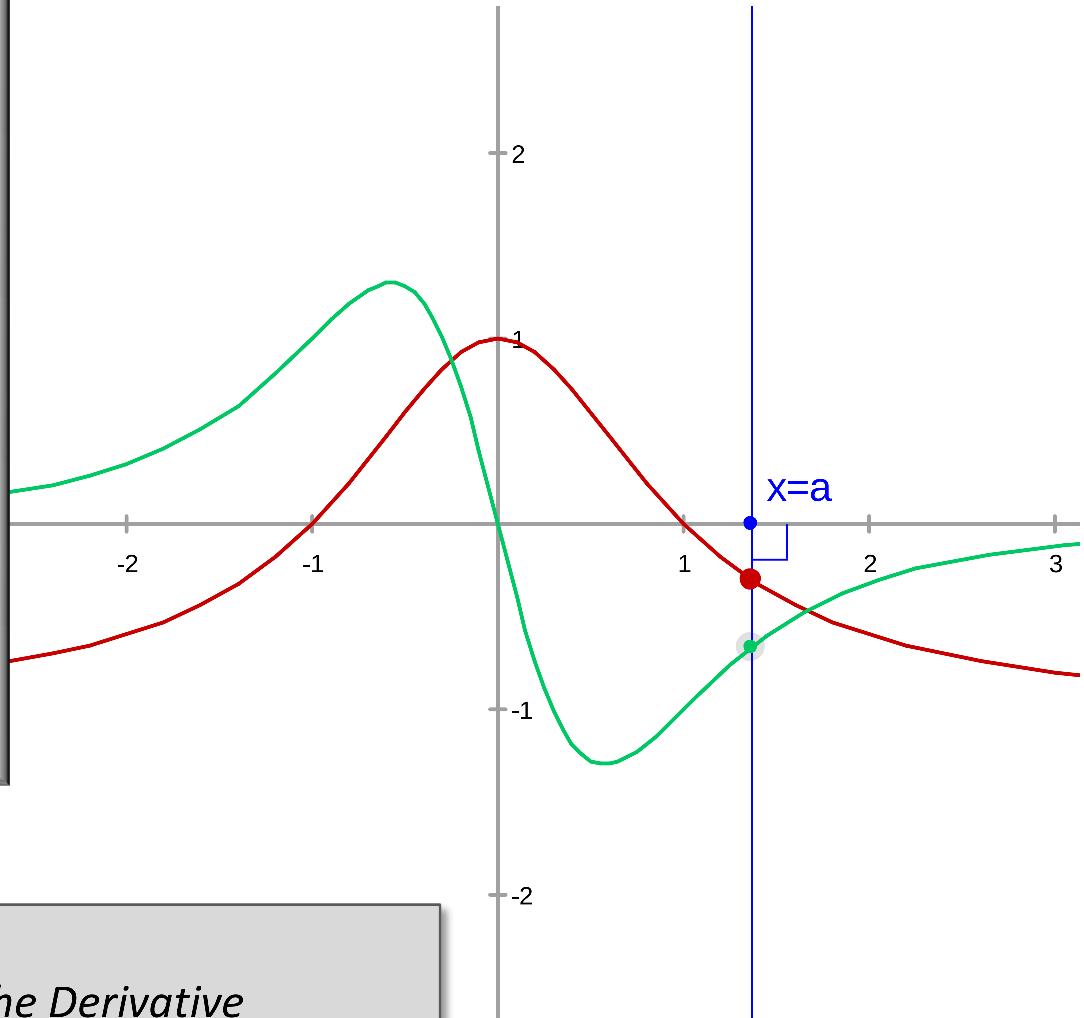
Learning Calculus With Geometry Expressions™

by L. Van Warren

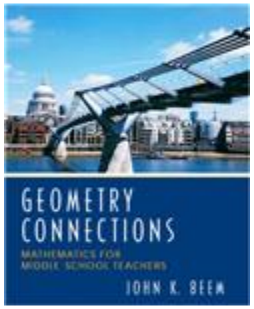


Lecture 10:
Definition of the Derivative

Chapter 3: Derivatives

LECTURE	TOPIC
10	**DEFINITION OF THE DERIVATIVE**
11	PROPERTIES OF THE DERIVATIVE
12	DERIVATIVES OF COMMON FUNCTIONS
13	IMPLICIT DIFFERENTIATION

Inspiration



John K. Beem (9)*

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Herbert Buseman (10)

Richard Courant (32)

David Hilbert (76)

C. Felix Klein (57)

Julius Plücker (1)

Christian Gerling (1)

Carl F. Gauß (8)

Professor John K. Beem Ph.D. USC 1968

His research interests included higher dimensional spaces and differential geometry.

He imparted to his students, an excellent sense of rigor on key concepts in calculus, particularly in defining and understanding the notion of limits.

He also won a $10,000 prize for teaching. When informed of this he smiled and then resumed his lecture.

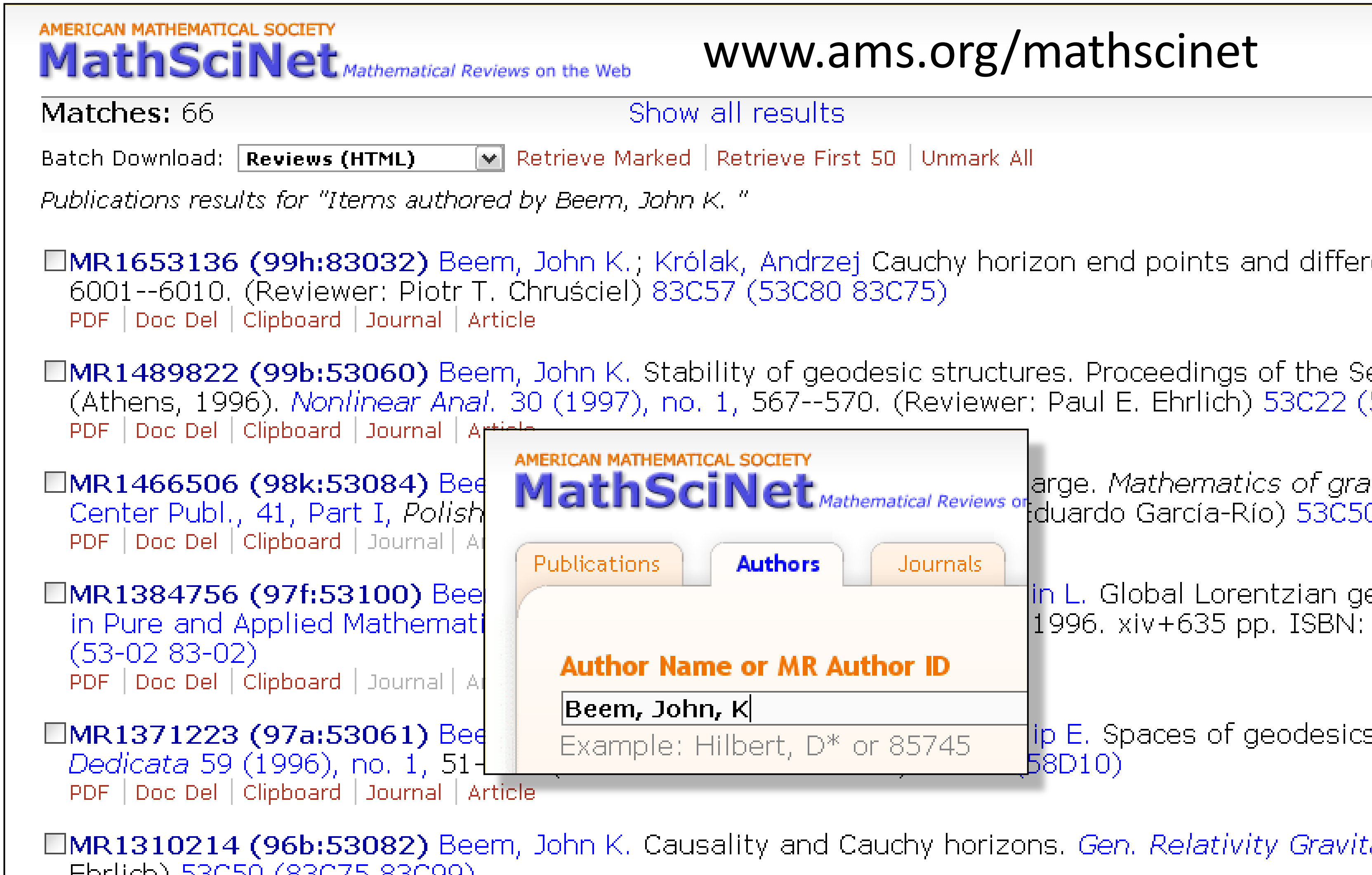
Shown here is Professor Beem's genealogy of mathematical mentors , going back to Carl Frederick Gauss in seven steps!

Regrettably no pictures of Prof. Beem or his advisor are available.

The parentheses show the number of students mentored to the Ph.D. level.

** Statistics courtesy Mathematics Genealogy Project*

Mathematical Publications of Professor Beem:



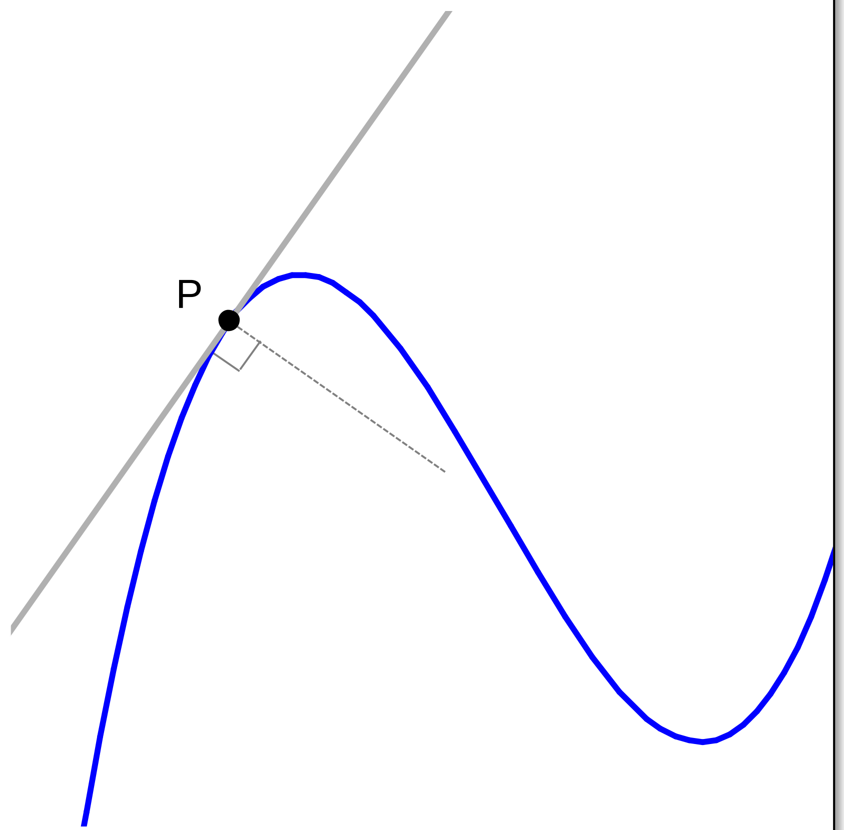
Lecture10-TangentLine.gx

Tangents

Consider some arbitrary curve that is smooth in some interval.

In that interval we can draw a line that lies along this curve at P.

This line is "tangent" to the curve at P.



EXERCISE:

1) Open the file. Drag the point P and notice how the slope changes as the line "rolls" along the curve

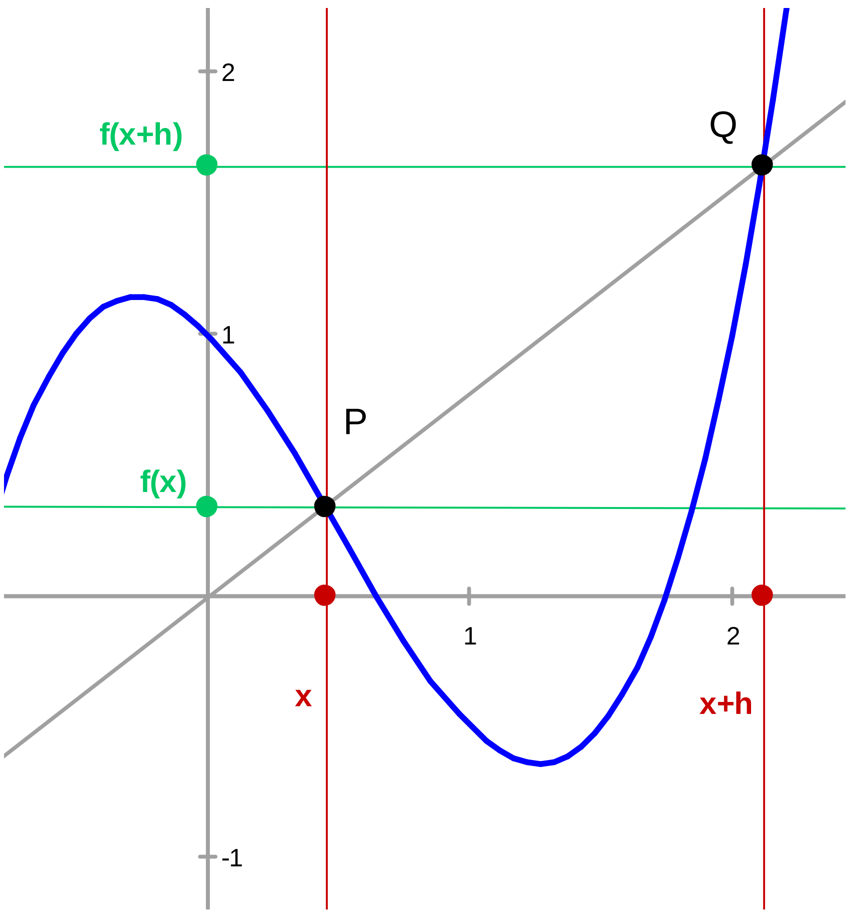
Lecture10-TangentConstruction.gx

Constructing the Tangent

Consider a function f(x) that is smooth in some x interval [a-h, a+h].

In that interval we can draw a line that connects points P and Q.

P is at (x, f(x)).
Q is at (x, f(x+h)).



EXERCISE:

1) Open the file. Drag each point. Think about the meaning of each.

Lecture10-TakeTheLimit.gx

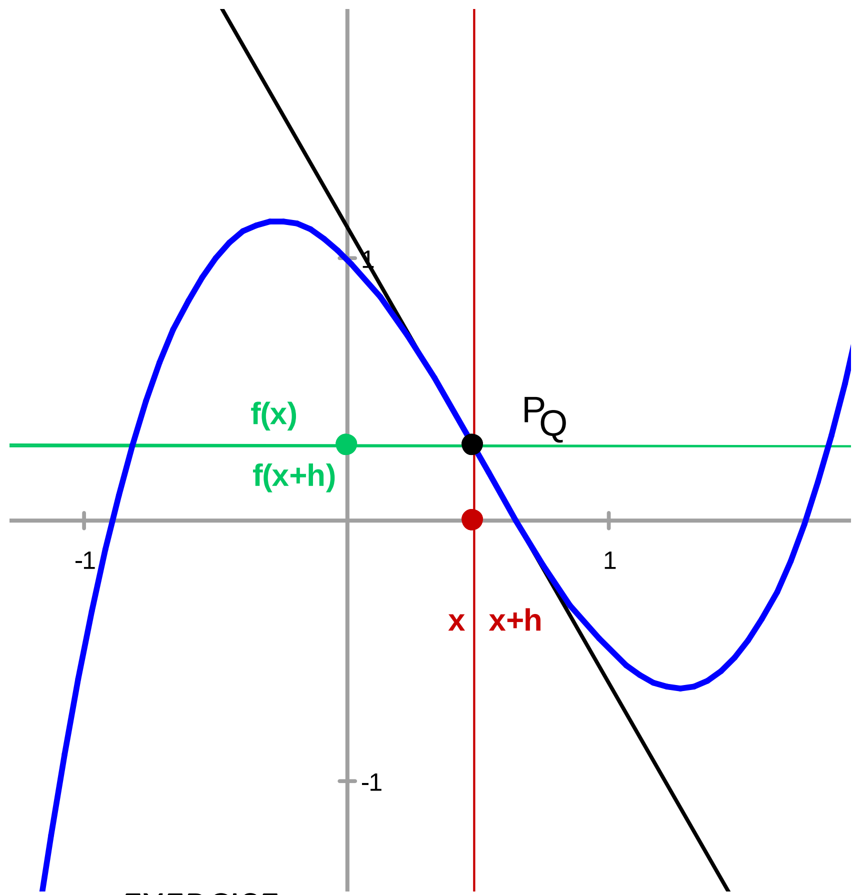
Taking the Limit

Now take the limit:

limit((f(x+h)-f(x))/h, h → 0)

P and Q become closer.

When they meet h = 0,
and the line is tangent
to the curve at P,Q.



EXERCISE:

1) Open the file. Move the line x+h towards the line x, making h=0.

Three Steps in Taking the Derivative

Steps:

1) Draw the function, *f(x)*
2) Draw two points, P and Q

   P = (x, f(x)), Q = (x, f(x+h))
3) Find the limit *f'(x)* using:

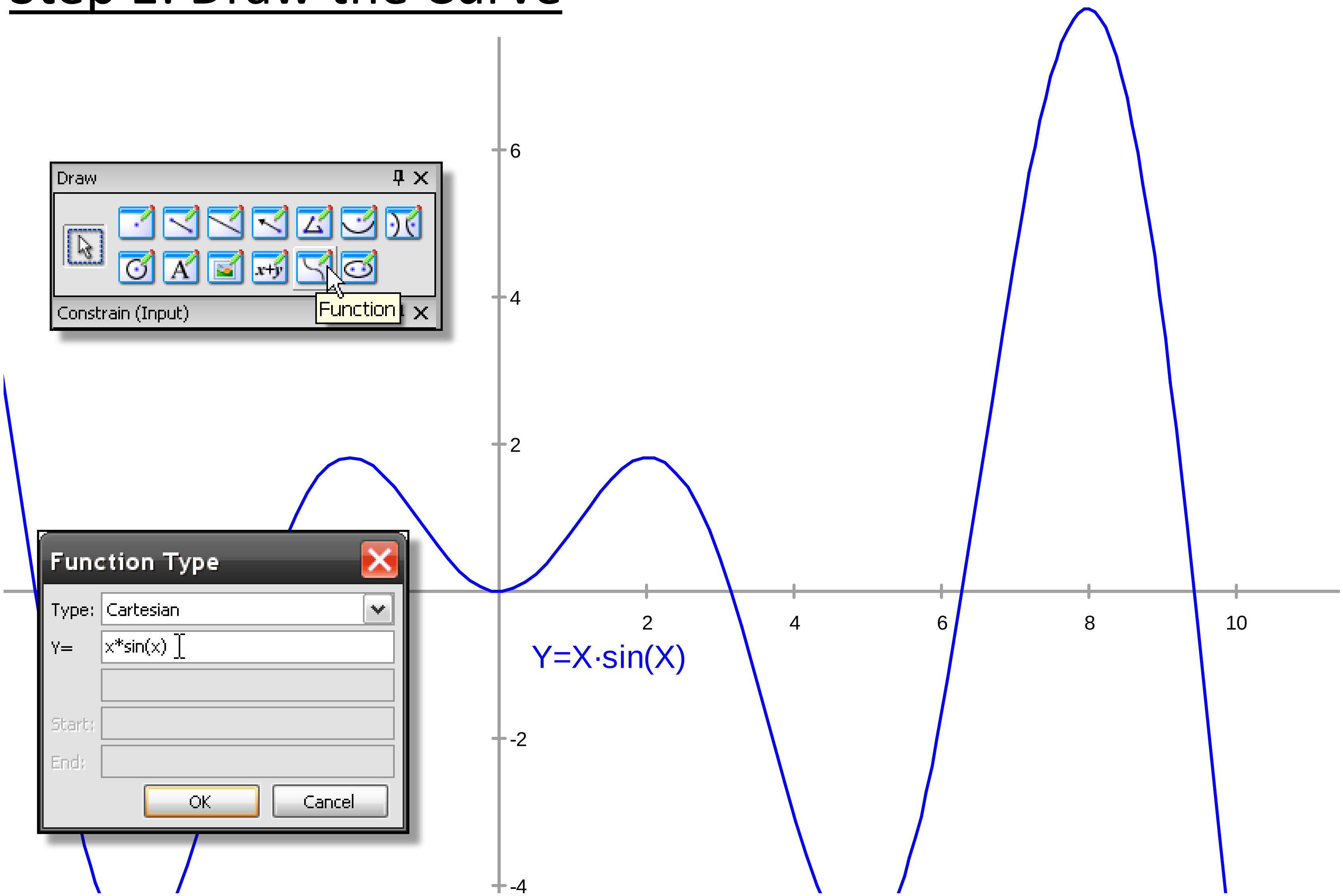
   f'(x) = limit((f(x+h)-f(x))/h, h → 0)

   also written as:

$$f'(x) = \frac{df}{dx} = \frac{lim}{h \to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

Lecture10-PickAFunction.gx

Step 1: Draw the Curve



Lecture10-PickAFunction.gx

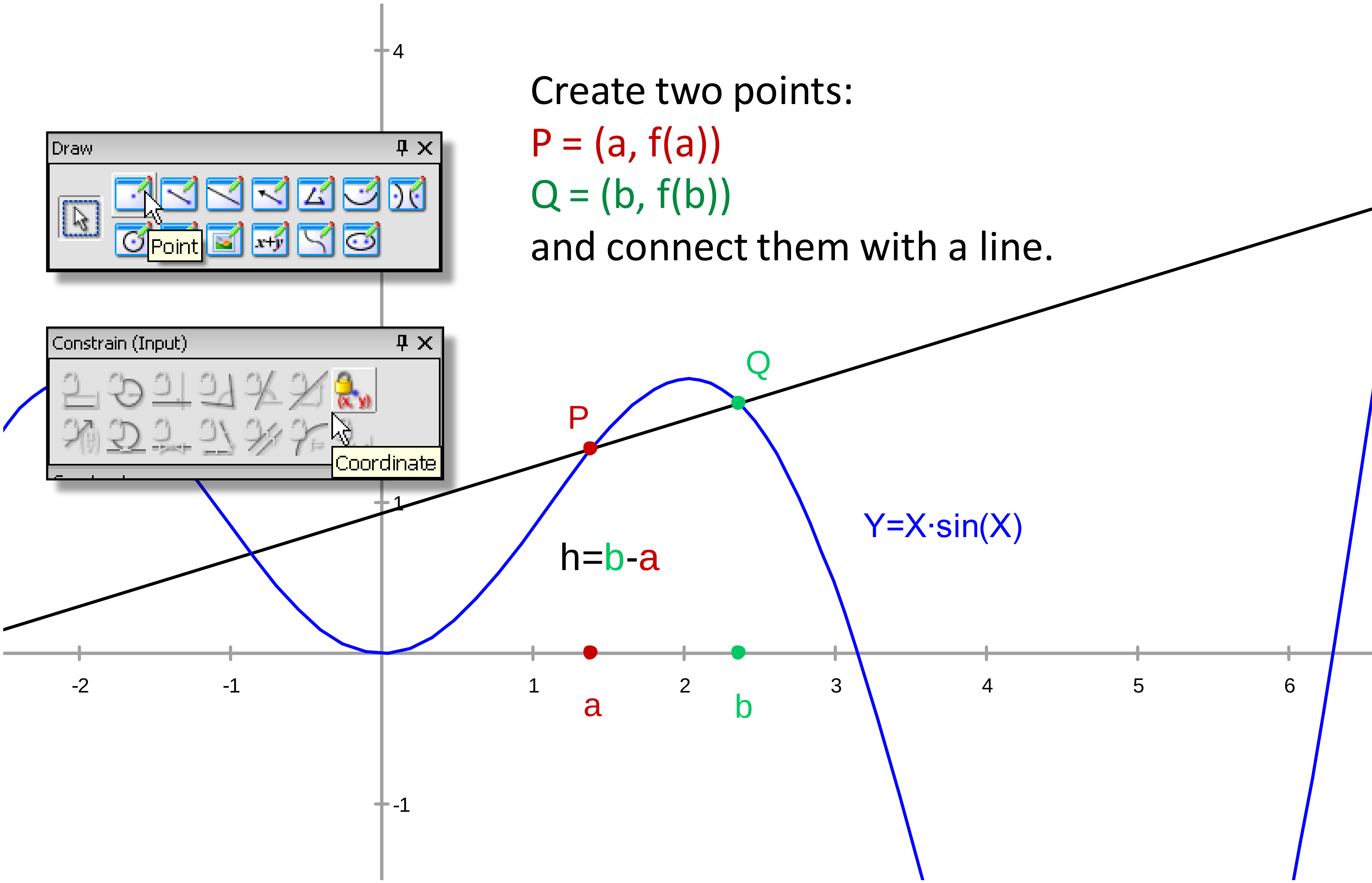
Step 2: Mark the Curve

Create two points:

P = (a, f(a))

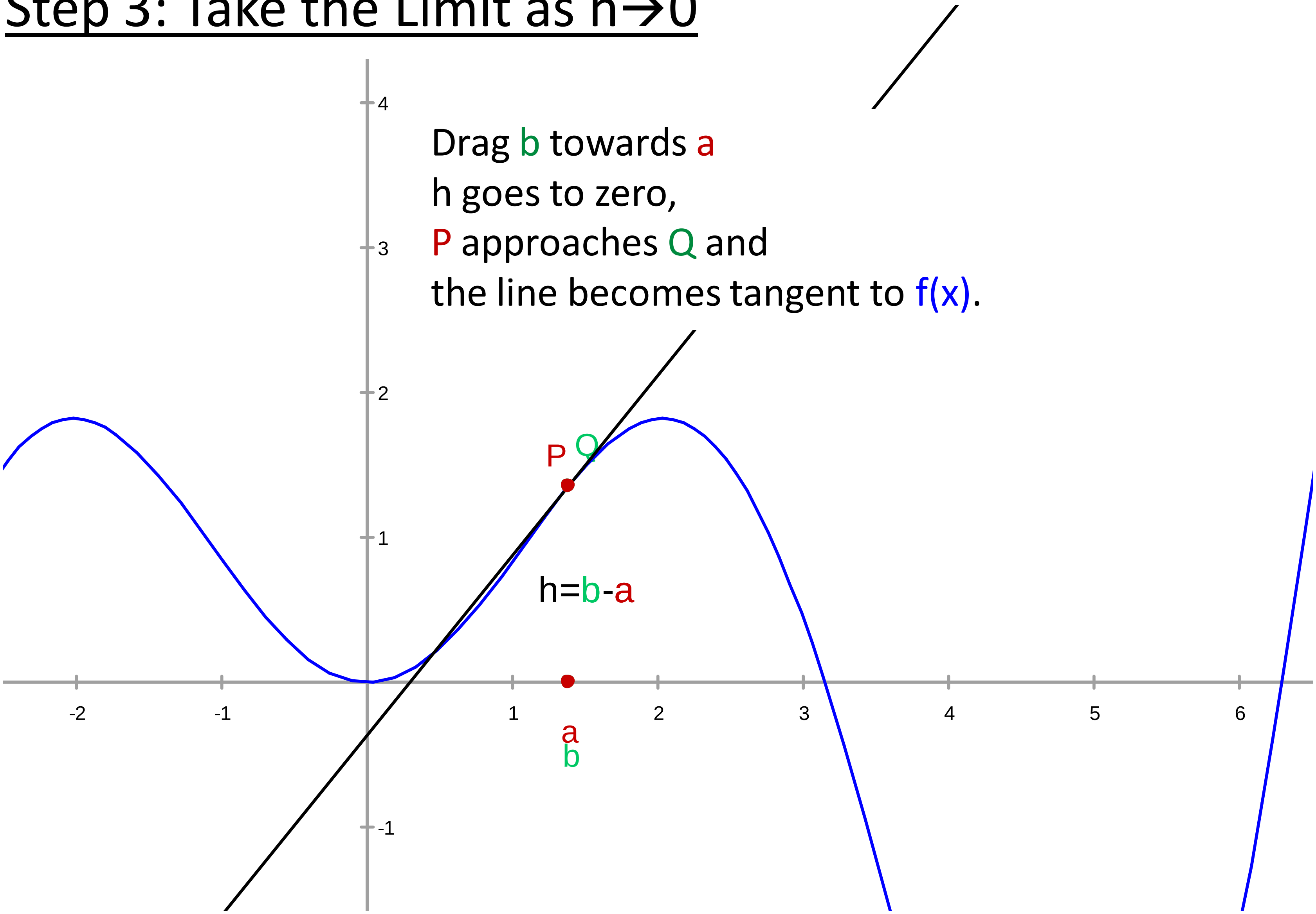
Q = (b, f(b))

and connect them with a line.



Lecture10-PickAFunction.gx

Step 3: Take the Limit as h→0



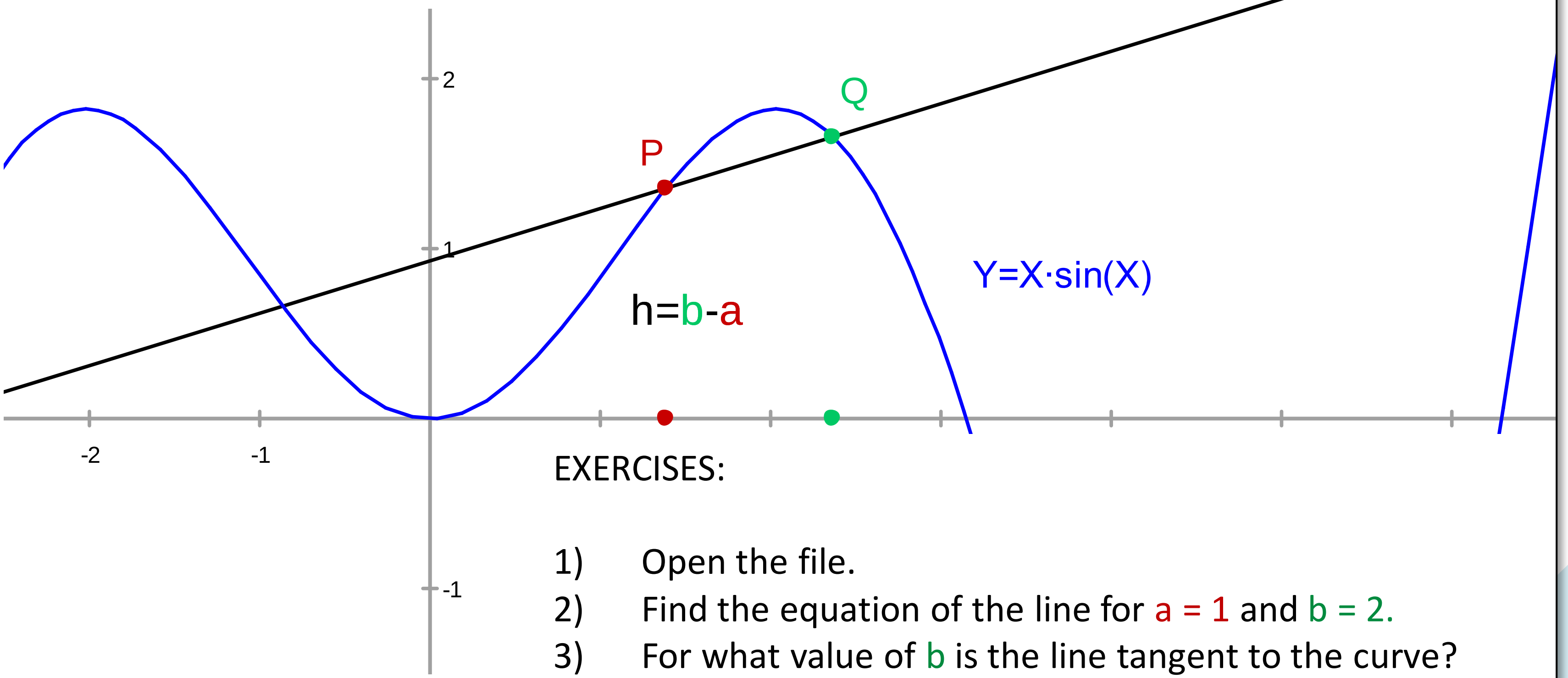
Lecture10-PickAFunction.gx

Equation of the Secant Line

The equation of a secant line is found using point-slope form:

$y - y_1 = m(x - x_1)$ → $y - f(a) = m(x - a)$

$m = (f(b) - f(a)) / (b - a)$ $y = m(x - a) + f(a)$



EXERCISES:

1) Open the file.
2) Find the equation of the line for $a = 1$ and $b = 2$.
3) For what value of b is the line tangent to the curve?

Find Derivative by Taking Limit

The derivative of f(x) is an equation that gives the slope of the line tangent to f(x) at a given x:

$$f(x);$$

$$x\sin(x)$$

$$f(x+h)-f(x);$$

$$(x+h)\sin(x+h)-x\sin(x)$$

$$\frac{f(x+h)-f(x)}{h};$$

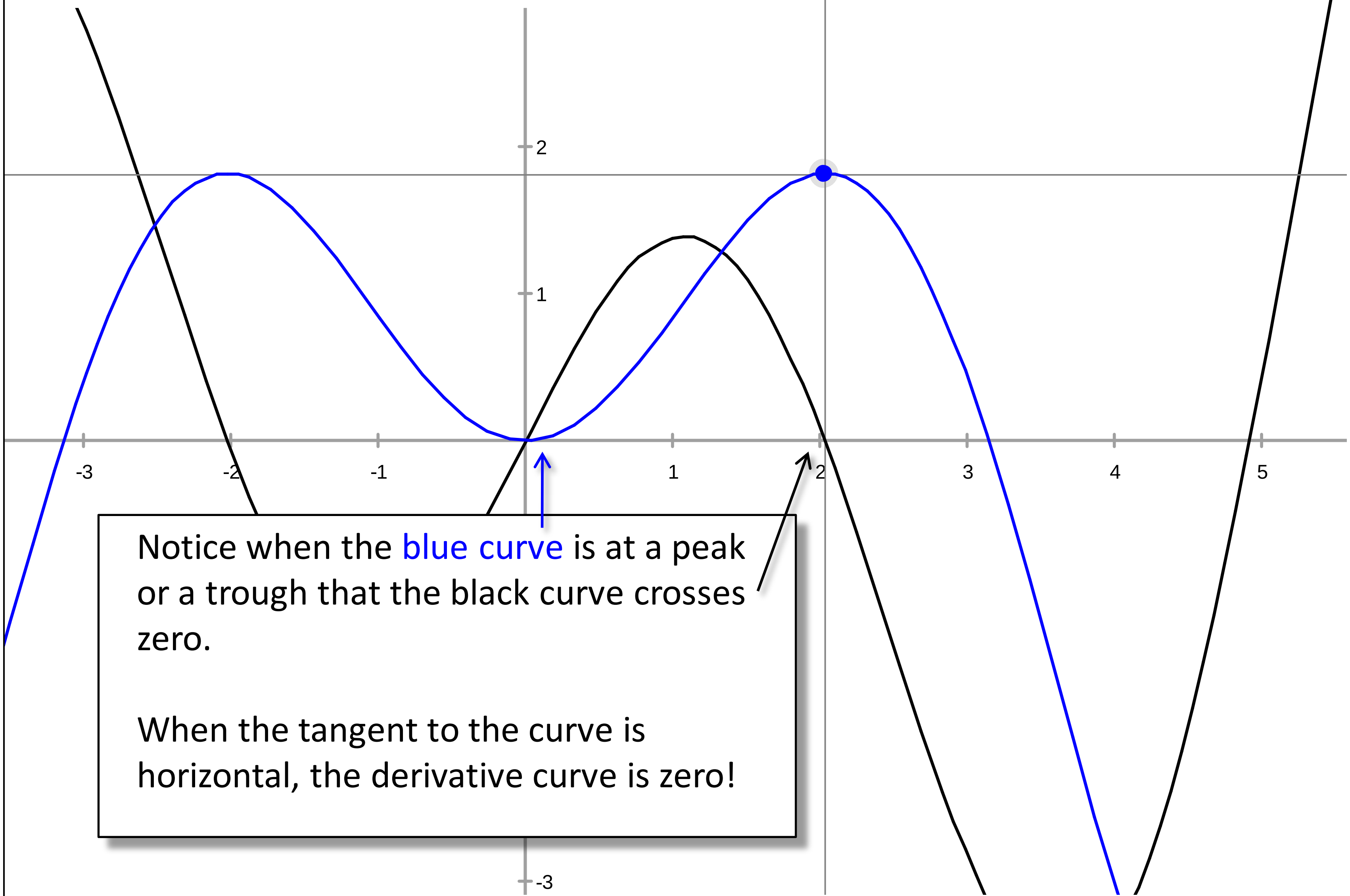
$$\frac{(x+h)\sin(x+h)-x\sin(x)}{h}$$

In the next lecture we will show that the limit of x sin (x) is using the product rule:

$$\cos(x)\,x+\sin(x)$$

The Derivative of a Function

Lecture10-FunctionAndDerivative.gx



Notice when the blue curve is at a peak or a trough that the black curve crosses zero.

When the tangent to the curve is horizontal, the derivative curve is zero!

Limit and Derivative Synonyms

Finding = Taking = Obtaining the Limit

Finding = Taking = Obtaining the Derivative =

Differentiating!

---

df/dx = f'(x) = f' all refer to the same idea
The Derivative!

Finding The Derivative of Square Root

$$\text{if } f = \sqrt{x} = x^{\frac{1}{2}} \text{ then } \quad \textit{find } f'$$

the rule will state

$$\frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}}$$

Take the Derivative Using the Definition

prove :

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{df}{dx} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$\frac{df}{dx} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$\frac{df}{dx} = \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$$

Take the Limit as *h* Becomes Vanishingly Small

$$\frac{df}{dx} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

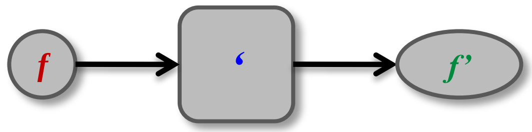
$$\frac{df}{dx} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$\frac{df}{dx} = \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$\frac{df}{dx} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}}$$

Derivative as Functional

A *functional* is a **function** that takes a function as input and produces a new function as output:



We saw earlier that the limit is a functional. Since the definition of the derivative is based on the limit, the derivative is also a functional.

$$f'(x) = \frac{df}{dx} = \frac{lim}{h \to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

Prime Notation

We denote the derivative of a function $f(x)$ by placing a tickmark (or "prime") after the name of the function. Thus, the derivative of $f(x)$ is $f'(x)$.

If we take the derivative again, we obtain the second derivative of $f(x)$ which is denoted $f''(x)$.

This process can continue indefinitely.

This notation is called LaGrange notation.

The Power Rule

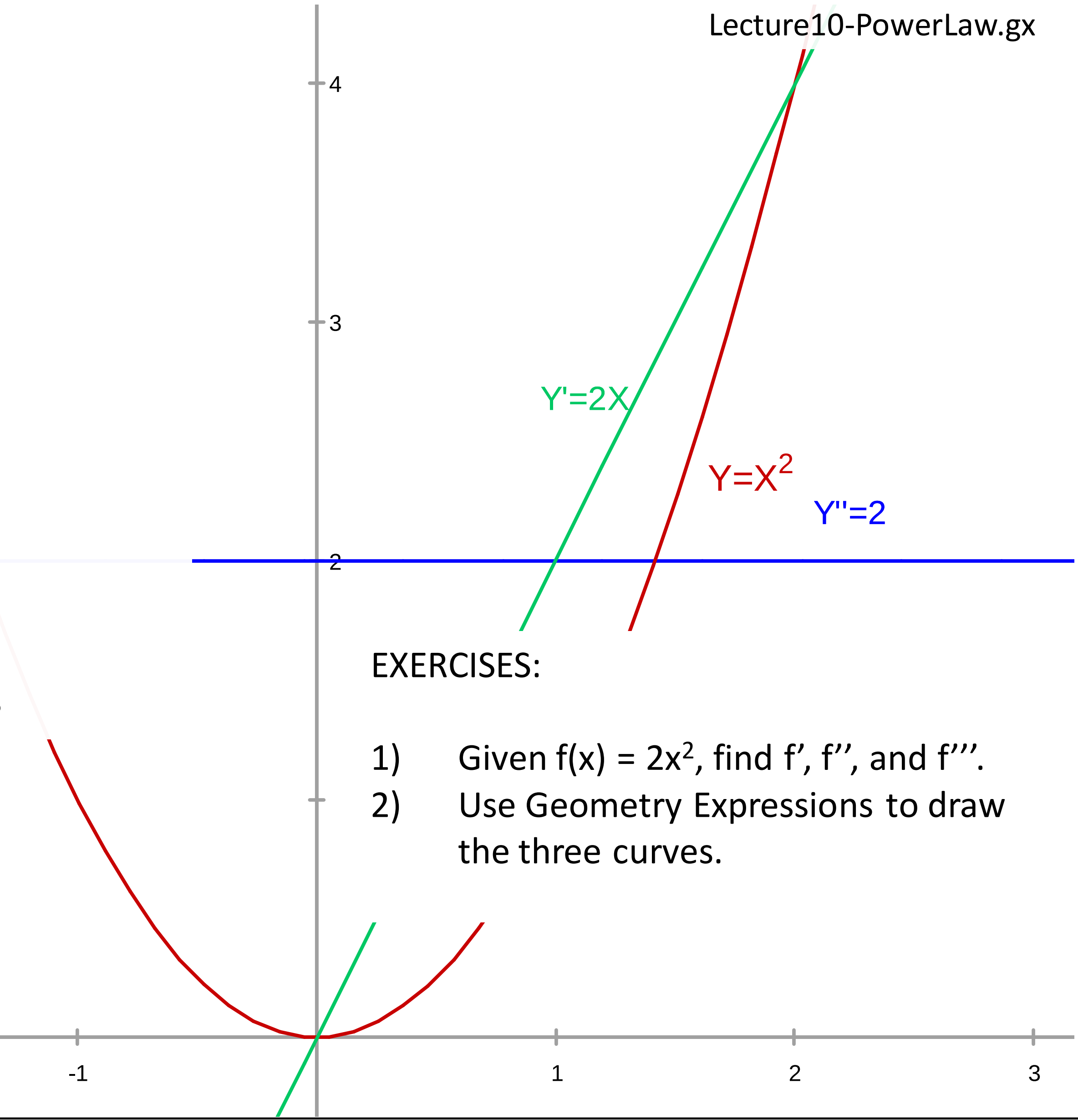
For a polynomial:

$f(x) = x^n$

The derivative is:

$f'(x) = nx^{n-1}$

Proof requires binomial theorem.



EXERCISES:

1) Given $f(x) = 2x^2$, find f', f'', and f'''.
2) Use Geometry Expressions to draw the three curves.

Definition of Derivatives

Given a function we can find its derivative, and this new function tells us the slope of a line tangent to the original curve.

We can also find the derivative of a function at one specific point. This is a number instead of a function.

This calls for an example!

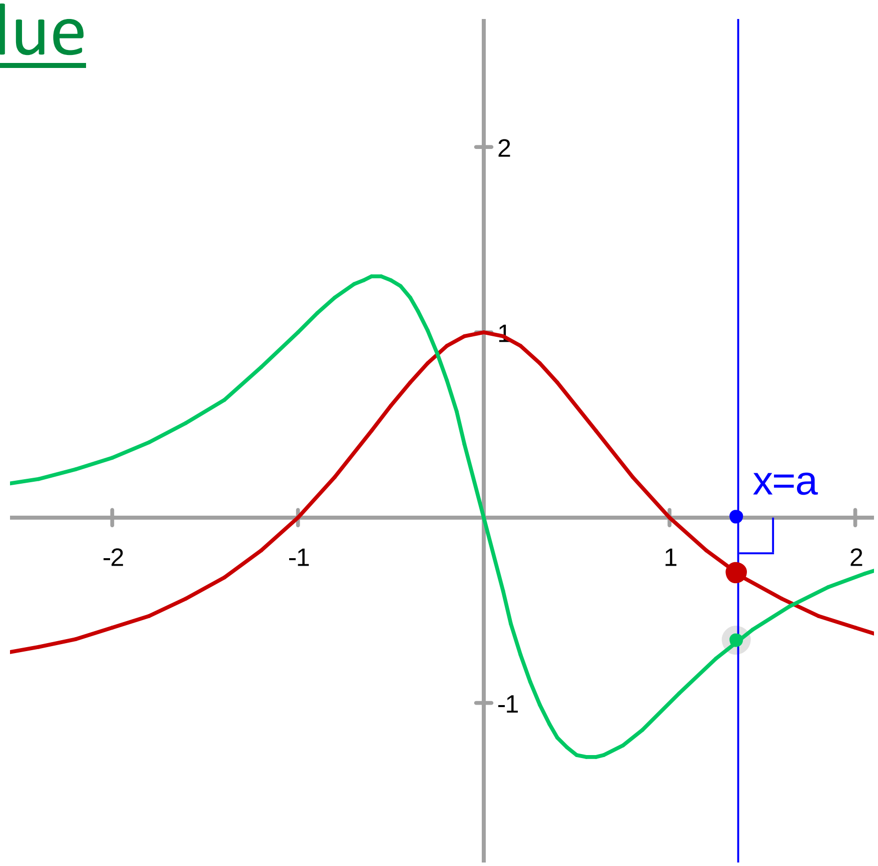
Lecture10-DerivativeCurveVsValue.gx

Derivative Curve Versus Value

The red curve is the original function.

The green curve is the derivative for any x.

The green dot marks the derivative for x = a.



EXERCISE:

1) Open the file. Drag the red point.
2) What functions give these curves?

Lecture10-DerivativeCurveVsValue.wxm

Symbolic Calculus

wxMaxima™ inputs and outputs

Just as we use the computer to help us with geometry, it can also help differentiate expressions too time-consuming to do by hand.

wxMaxima™ is a free program that does symbolic calculus.

```
(%i1) (x^2-1)/(-x^2-1);
```

(%o1) $\dfrac{x^2-1}{-x^2-1}$

```
(%i2) diff( (x^2-1)/(-x^2-1),x);
```

(%o2) $\dfrac{2\,x\left(x^2-1\right)}{\left(-x^2-1\right)^2}+\dfrac{2\,x}{-x^2-1}$

EXERCISE:

1) Plot these curves using Geometry Expressions™.
2) Do they look familiar?

The Derivative For Any *x*

The derivative of the function $y = f(x)$ is $f'(x)$:

$$f'(x) = \frac{dy}{dx} = \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

For any x, provided the limit exists at x.

The Derivative For A Specific *x*

The derivative f ' (x) evaluated at x = a is:

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

For any a, provided the limit exists at a.

Lecture10-PolynomialRoots.gx

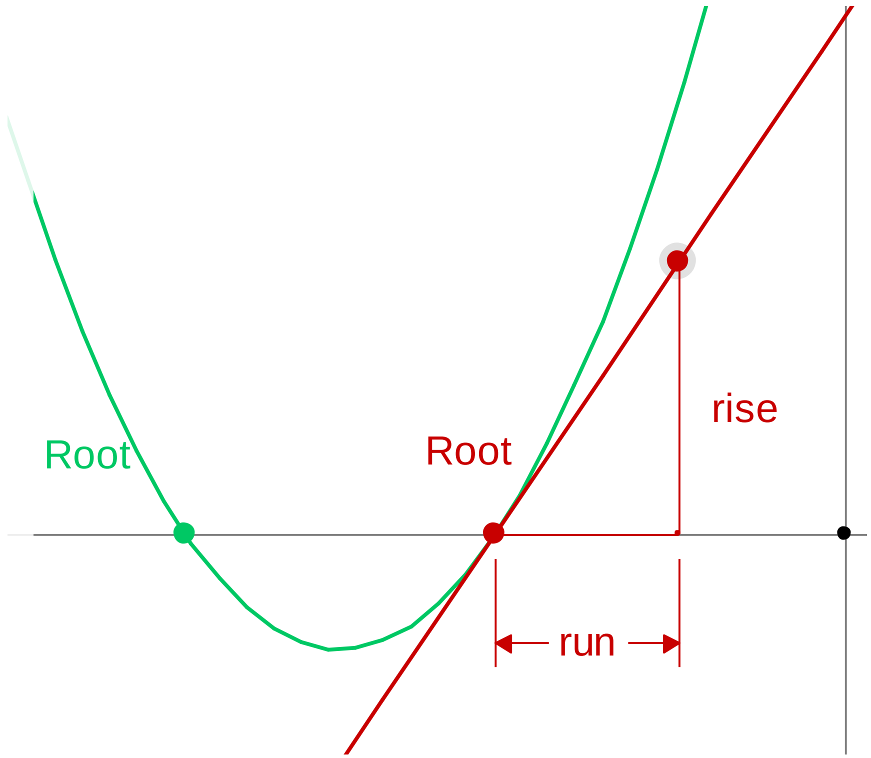
The Roots of A Polynomial

Consider y = f(x).

The x-axis crossings where y = 0 are called the "roots" of f(x).

They are also called the "solutions" of f(x).

They represent values of x for which y = f(x) = 0.



EXERCISE:

1) Open the file. Drag each point.
2) Write the equation for the line.
3) Write the equation for the curve.
4) Write the equation for the roots.
5) Click View→Show All to check.

Roots Versus Derivatives

$f(x) =$	some polynomial
$f(x) = 0$	"solutions" or "roots" of the polynomial
$f'(x) =$	derivative / slope / rate of the polynomial f(x)
$f'(x) = 0$	local minima or maxima why?
	slope is only zero at crest or trough of f(x)

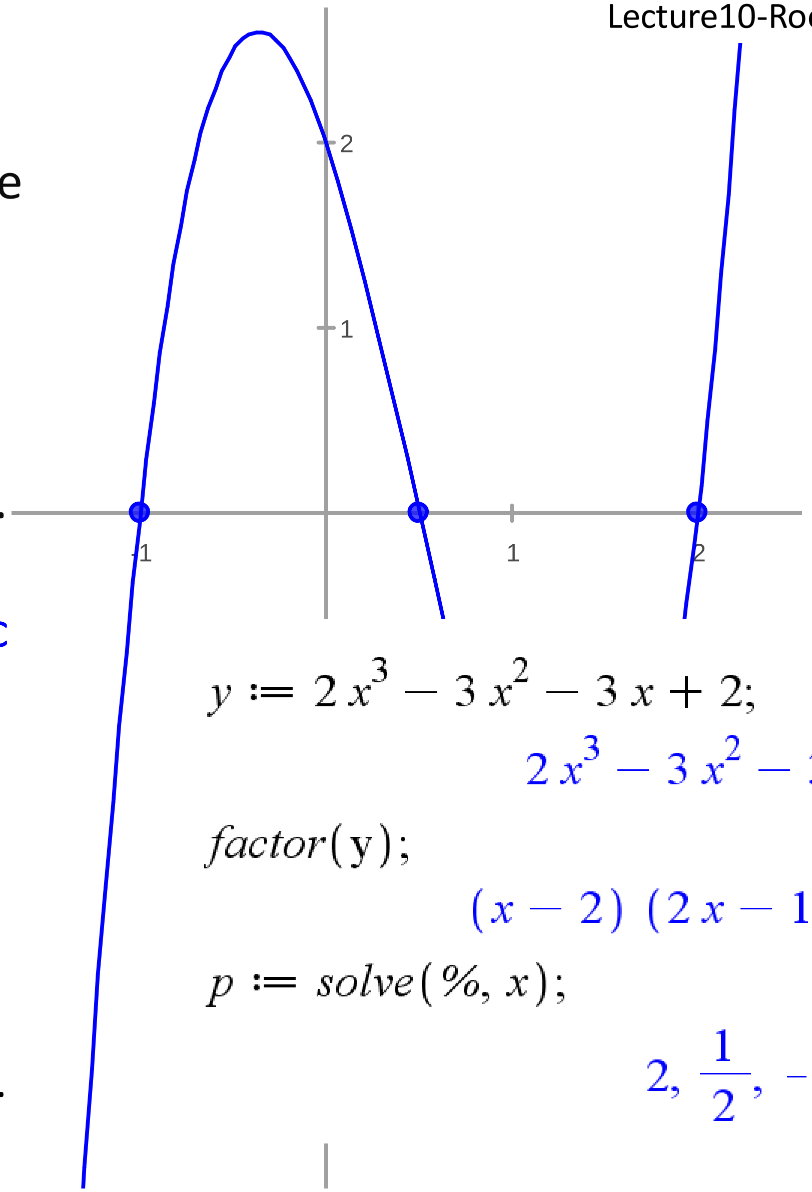
Roots Of A Cubic

Lecture10-RootsOfACubic.gx

Maple™ is a program that, like wxMaxima™ does symbolic calculus.

It is not free, but is available with an educational discount.

It was used to solve this cubic equation.



$$y := 2\,x^3 - 3\,x^2 - 3\,x + 2;$$

$$2\,x^3 - 3\,x^2 - 3\,x + 2$$

$$factor(y);$$

$$(x-2)\,(2\,x-1)\,(x+1)$$

$$p := solve(\%, x);$$

$$2, \frac{1}{2}, -1$$

EXERCISE:

1) Open the file.
2) Click View→ Show All
3) Change the equation slightly.
4) Find and draw the new roots.

Derivative Of A Cubic

Lecture10-DerivativeOfACubic.gx

The roots of the derivative curve show the peaks and valleys of the original. We will study this more in “max-min” theory.

Here we use Maple™ to find the derivative and roots of original function:

$$print(f);$$

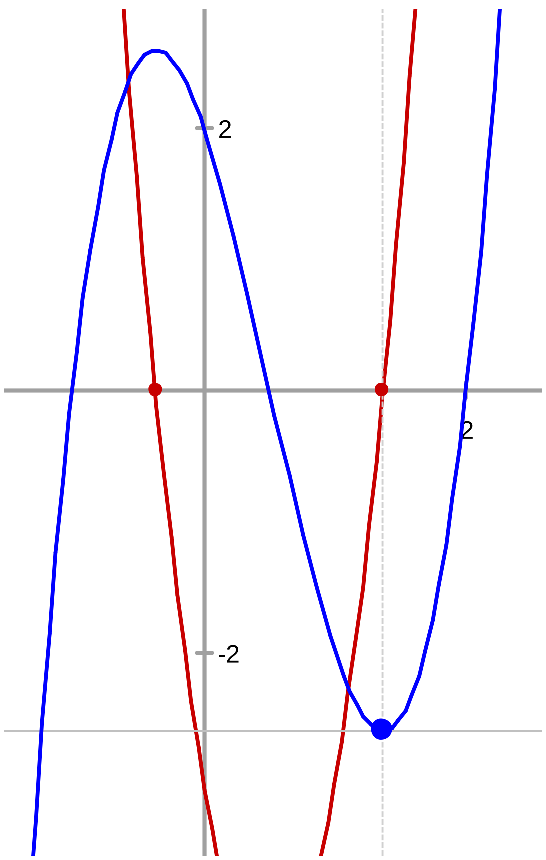
$$x \rightarrow 2x^3 - 3x^2 - 3x + 2$$

$$fprime := x \rightarrow 6x^2 - 6x - 3;$$

$$x \rightarrow 6x^2 - 6x - 3$$

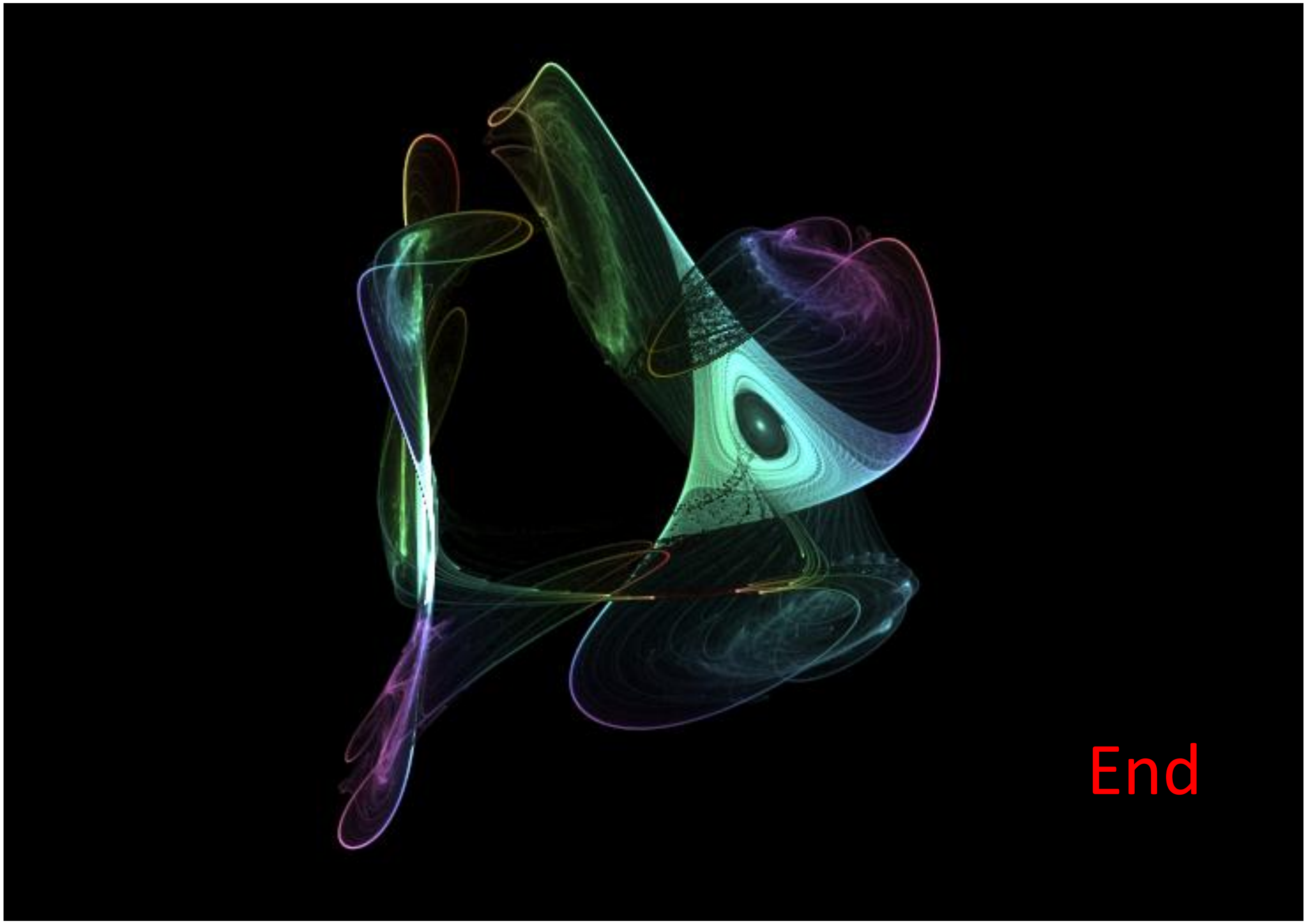
$$evalf(solve(6x^2 - 6x - 3, x));$$

$$1.36,\ -0.365$$



EXERCISES:

1) Open the file, Click View→ Show All
2) Note the hairline guides in the “valley”.
3) Construct an equivalent set for the “peak”.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