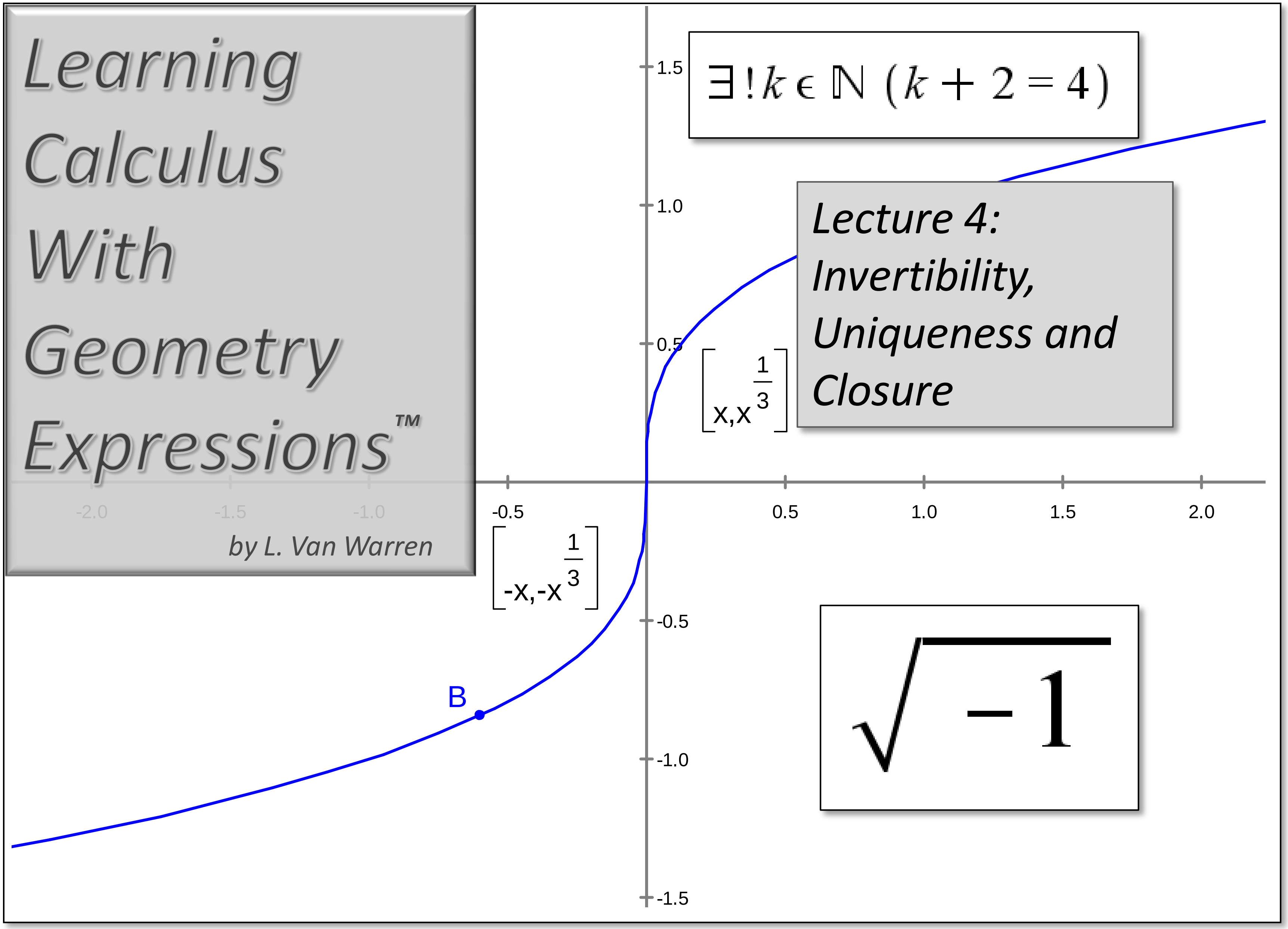
Learning Calculus With Geometry Expressions™

by L. Van Warren

Lecture 4: Invertibility, Uniqueness and Closure

$$\exists ! k \in \mathbb{N} \; (k + 2 = 4)$$

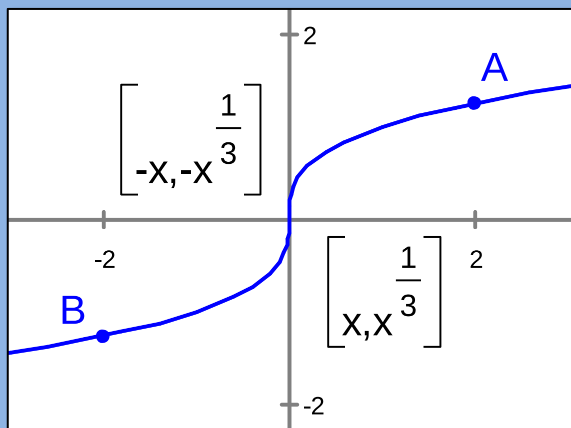
$$\sqrt{-1}$$



Chapter 1: Functions and Equations

Lecture	*Topic*
0	*Geometry Expressions™ Warm-up*
1	*Explicit, Implicit and Parametric Equations*
2	*A Short Atlas of Curves*
3	*Systems of Equations*
4	***Invertibility, Uniqueness and Closure***

Invertibility:



$y = x^3$ is invertible
$y = x^2$ is not. Why?

Uniqueness:

$$\exists ! k \in \mathbb{N} \ (k + 2 = 4)$$

There exists one natural number k such that k+2 = 4.
Is this true?

Closure:

$$\sqrt{-1}$$

The square root
of a negative real isn't.

- *Lynn Warren*

The author with **Ron Mallett, Ph. D.** who solved a special case of Einstein relativity equations regarding time travel. Mallett designed a time machine based on a ring laser.

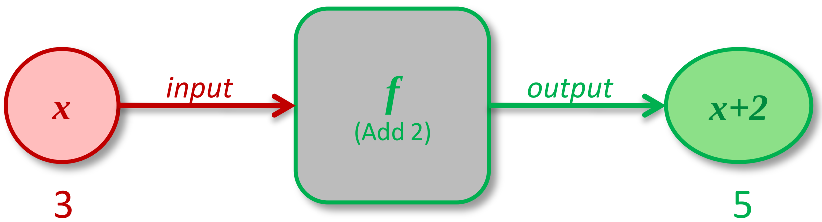
Nicolas Gisin, Ph. D., Geneva
Did Gisin build Ron Mallett's time machine? Gisin demonstrated "action-at-a-distance" using coupled photons. Einstein called this "spooky" but is it just a case of the twins paradox?

Invertibility

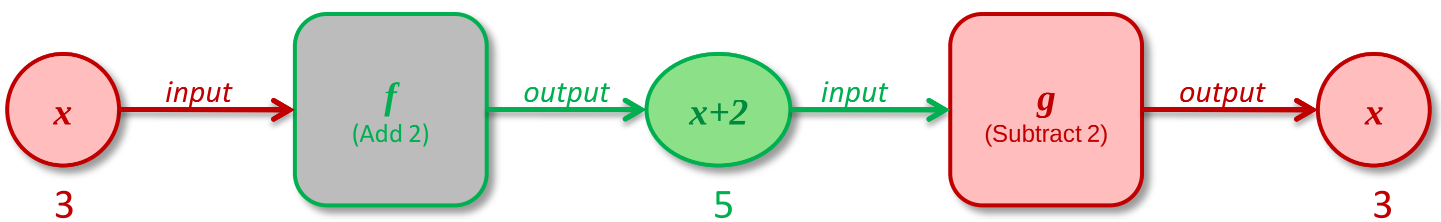
This is a simple idea to consider, and yet has far-reaching consequences.

When we say that a function ***g( )*** is the ***inverse*** of a function ***f( )***, we mean that ***g( ) undoes*** the effect of f***( )***.

Consider the function machine ***f*** representing $f(x) = ((x)+2)$, with its inputs and outputs:



Or the function machine ***g***, shown to the right, busily undoing the effects of ***f*:**

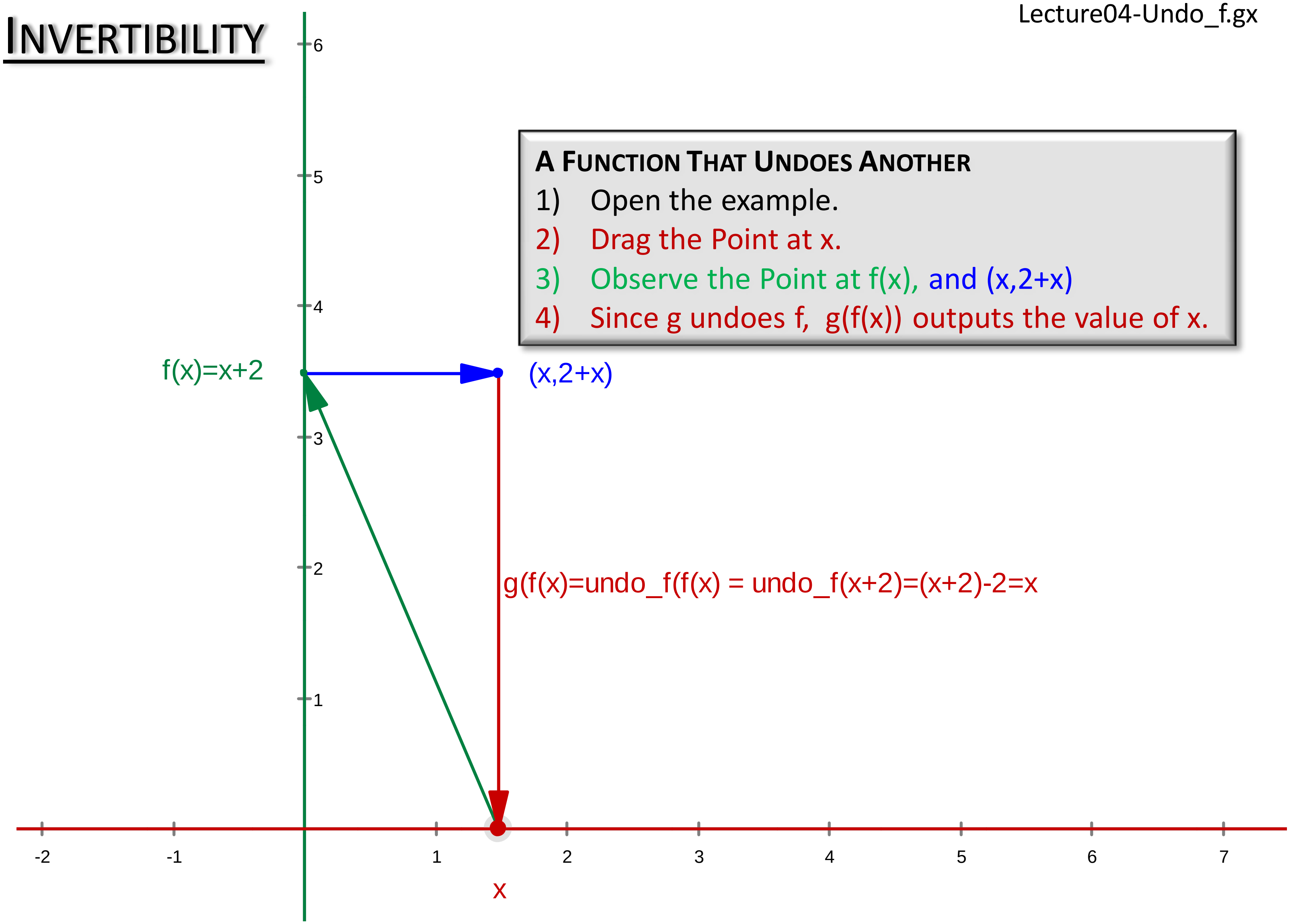


INVERTIBILITY

Lecture04-Undo_f.gx

A FUNCTION THAT UNDOES ANOTHER

1) Open the example.
2) Drag the Point at x.
3) Observe the Point at f(x), and (x,2+x)
4) Since g undoes f, g(f(x)) outputs the value of x.



What's in a Name, Even a Dummy Name?

You may have noticed that I *finessed* a certain issue.

I said what f was and gave the symbolic expression $f(x) = ((x)+2)$. A geometry expression followed. I did not write $g(x) = ((x) - 2)$. I did not want to reuse the symbol x since x was in the *(scope)* of f.

Know this; The x in $g(x) = ((x) - 2)$, is a DIFFERENT x than the x of $f(x) = ((x)+2)$. Another way of saying that the x is different is to use a different name for it. Color was used here to make the point. Names can be used to make this distinction, but color is better. ***Color*** talks to the brain's visual processing system, while ***names*** talk to the auditory processing system. Using a different name for a similar input is called a ***dummy variable.*** The need to make up fake names on-the-fly appears often in calculus. Alonzo Church invented a special calculus called the ***lambda calculus,*** the calculus of as few names as possible. If deprived of color we just say, $g(u) = u - 2$, to prevent confusion about the *scope*, or *persistence* of x.

$$x \longrightarrow \boxed{\begin{matrix} g \\ \text{sub } 2 \end{matrix}} \longrightarrow x\text{-}2$$

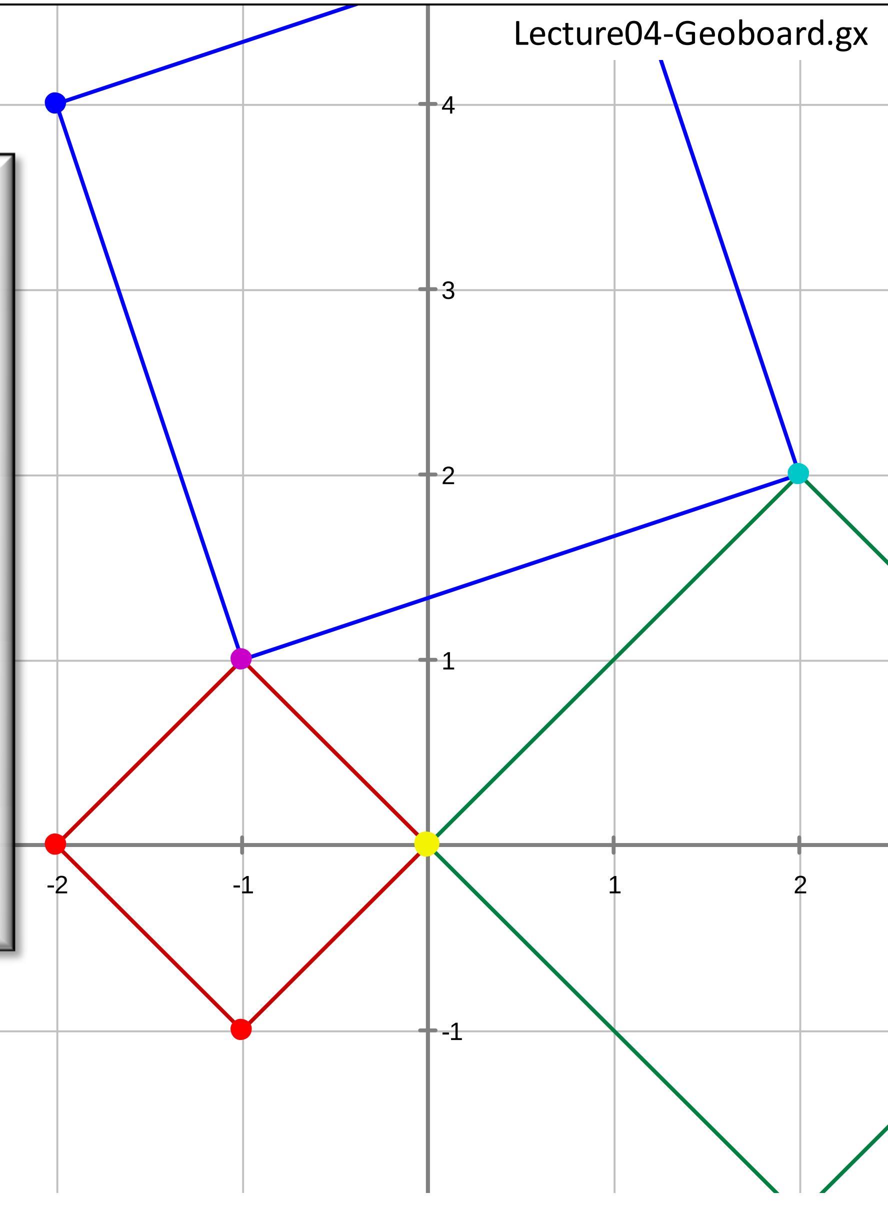
THE GEOBOARD

1) The Geoboard was [invented](#) by the blind mathematician Nicholas Saunderson (1682-1739), professor of mathematics at Cambridge.

2) What is the correct color for the missing (clipped) vertices on the blue and green squares?

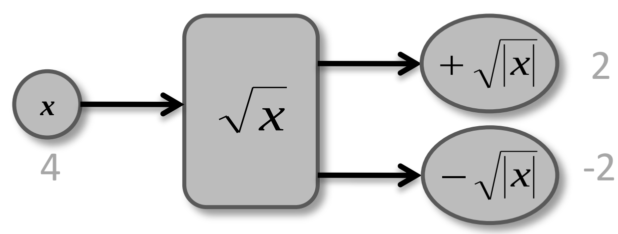
3) What scheme is being used to determine vertex coloring?

4) How might this ***color algebra*** be extended to higher dimensions?



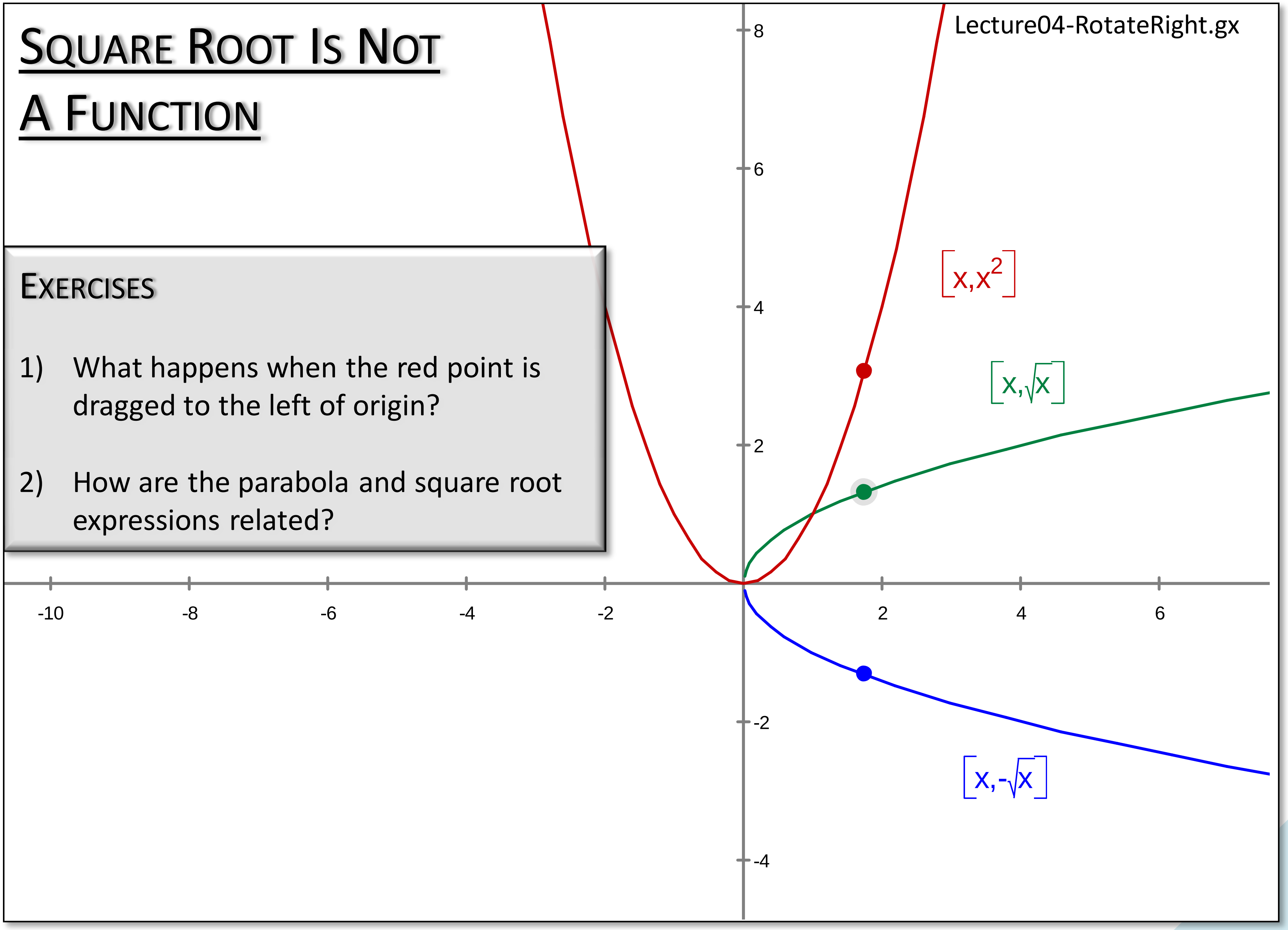
Functions vs. Equations: Count Possible Solutions

If we get ***more than one output*** from an expression we say the expression is an ***equation*** *or more generally, a* ***relation***. A famous example of this is ***square root***. The square root of an integer like 4, has two possible integer square roots in the integers, 2 and -2. Whenever you solve an equation where you take the square root, remember that it has two possible solutions. Therefore square root, or any equation that has more than one solution or output, cannot be a function.



Square Root is not a function.

Square Root Is Not A Function



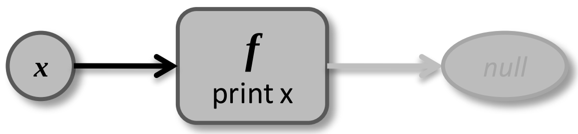
Exercises

1) What happens when the red point is dragged to the left of origin?

2) How are the parabola and square root expressions related?

Functions That Are Null

If we get **no output for any input** we say the function is the ***null function***, an oddity for sure, but necessary for completeness, if not Kurtness. I have a printer that produces no output.



Reflection Across Y = X Yields Inverse

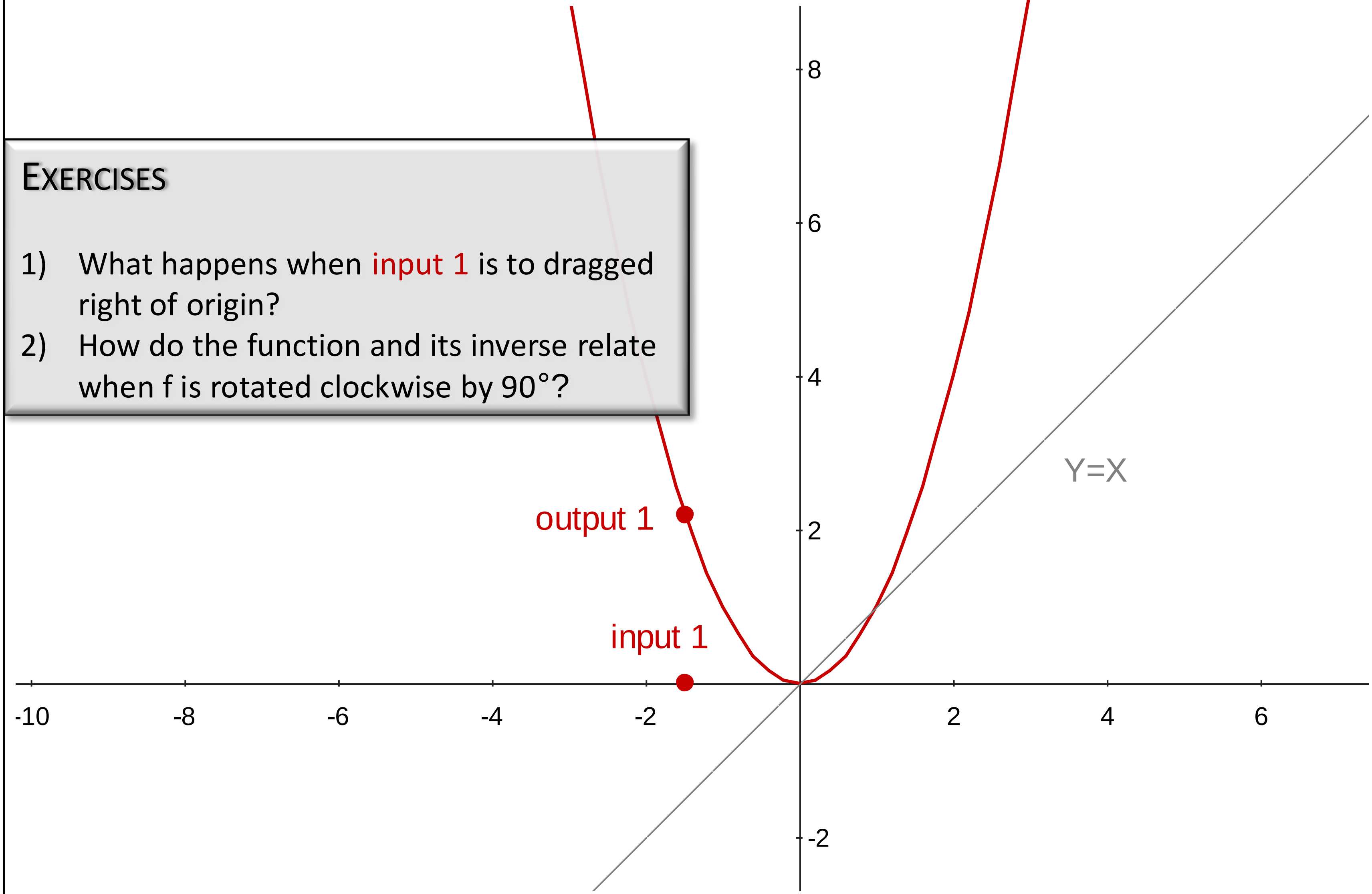
To obtain the function g that inverts a function f, reflect the graph of f across the line y = x.

To perform this symbolically for a function of y in terms of x you solve the equation so that x is in terms of y. If this solution does not exist, then the equation is not invertible. If the equation is not invertible it is not a function.

For some symmetric functions inversion is equivalent to a clockwise rotation of the function.

Reflection Across Y=X Yields Inverse

Lecture04-InputsAndOutputs.gx

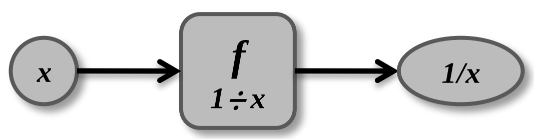


Exercises

1) What happens when input 1 is to dragged right of origin?
2) How do the function and its inverse relate when f is rotated clockwise by 90°?

Functions That Are Conditionally Invertible

We might have a function ***f*** that we can invert for some inputs but not others. If we specify which inputs for which ***f*** is invertible, then we have qualified the circumstances under which ***f*** is invertible. That is, we have provided those limitations that must apply for ***f*** to be invertible. If you're following this, and I know that you are, you see what a powerful idea this. We would call this, **conditionally invertible**. We would then have to worry about the intervals over which the function was invertible and how those intervals connect to each other. If a function is conditionally invertible we must provide these conditions and in the rigorous case, a proof. Consider for example $f(x) = 1/x$. This function is invertible provided that x doesn't equal zero.

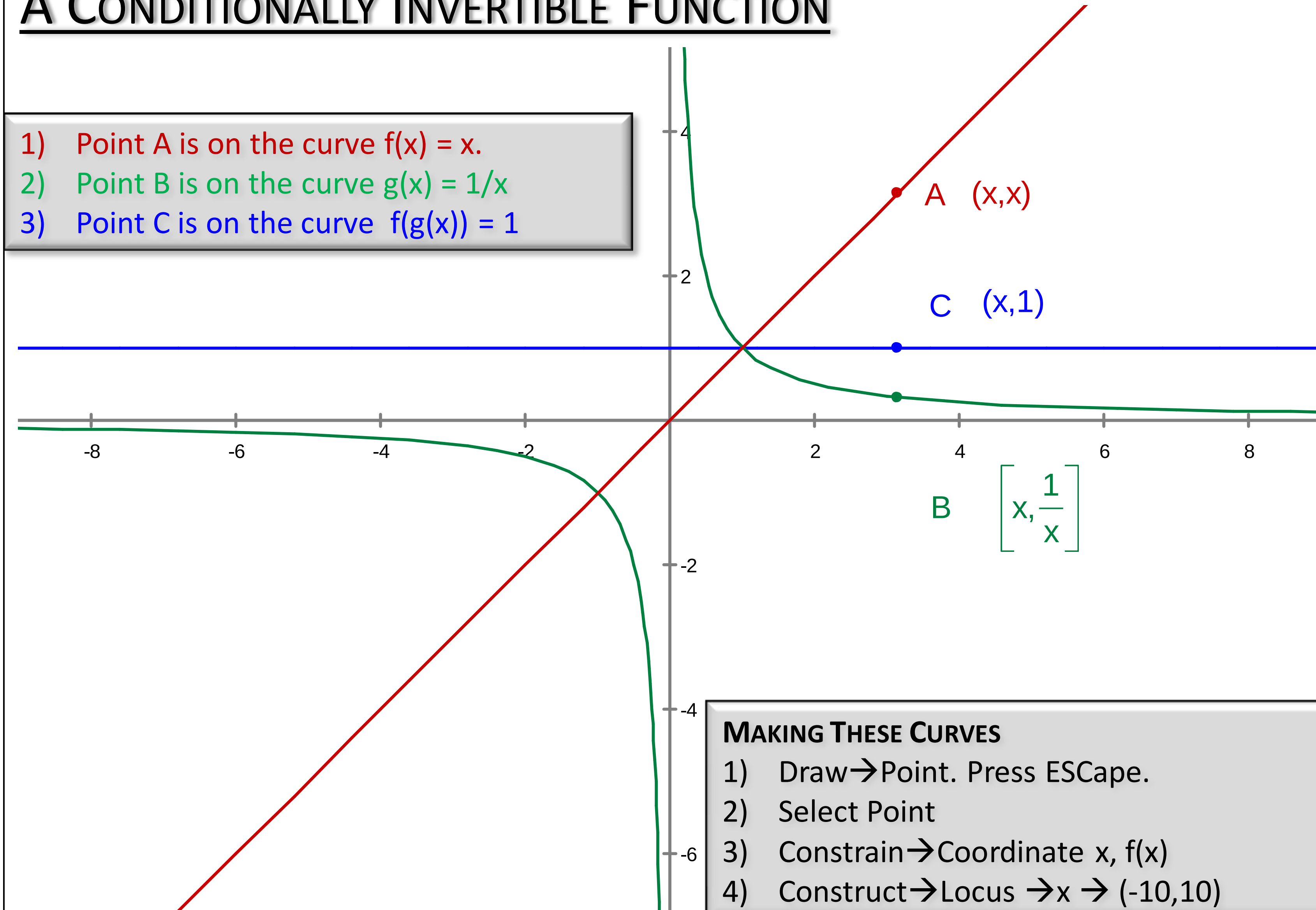


f(x) = 1/x is a function except when x = 0

A Conditionally Invertible Function

Lecture04-InvertibleX.gx

1) Point A is on the curve f(x) = x.
2) Point B is on the curve g(x) = 1/x
3) Point C is on the curve f(g(x)) = 1



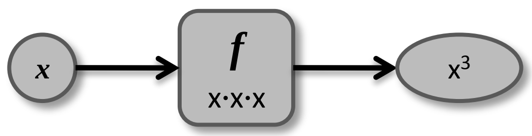
Making These Curves

1) Draw→Point. Press ESCape.
2) Select Point
3) Constrain→Coordinate x, f(x)
4) Construct→Locus →x → (-10,10)

INVERTIBILITY: INPUTS AND OUTPUTS TELL THE TALE

The inverse of $f(x)$ is often written $f^{-1}(x)$ where the -1 power does not represent the reciprocal $1/f(x)$, but rather the analogy of inverting, reversing or undoing the effect of $f(x)$.

A great deal of effort has gone into working out the issues that are clarified by the use of inputs and outputs. Long story short, an equation that produces ***one output for every input*** is a ***function***.



Up till now we have focused on functions that take one input and produce one output. If we have more than one output, the operation is not a function. Consider the converse: If an operation has more than one INPUT, it is not invertible either. Far reaching are these consequences. Consider the four basic operations of mathematics, addition, subtraction, multiplication and division. They are all "binary" operators, that is, they have two inputs. It is impossible to know which of an infinite combination of inputs produce a given output unless we save state and remember everything that was done along the way.

For example 2 + 2 = 4, but so does 1 + 3, and -2 + 6, and so on. If all we keep is the result then we lose information. When we lose information we incur undecideability, and the knowledge of what brought us to that state.

Conditional Invertibility and 1/x

The equation $y = 1/x$ is defined everywhere except the origin. It is conditionally invertible as the previous example demonstrated.

Traditionally we say that, at the origin, $1/x$ is "undefined".
But actually $1/x$ is well-defined at the origin, the trouble is, it has two values or outputs there, instead of one and the value depends on how we got there.

If we approach from the left hand side, the negative x-axis, from values that were much less than zero to values that were nearly zero, we would notice the value was $-\infty$.

If we came at the function from the right, from values that were much greater than zero to values that were nearly zero, we would notice that the value was $+\infty$.

Exactly at the origin, $1/x$ has two values. These values are $+\infty$ and $-\infty$. These values are a little frustrating to mathematicians because they couldn't be more different (they would say "divergent"). Also $-\infty$ and $+\infty$ are not specific values. There are different sizes of ∞ and that compounds the problem.

But $1/x$ is really just the definition of division. Division is trying to tell you something. At the origin it becomes two things instead of one and these two things couldn't be more different. I'll leave it at that.

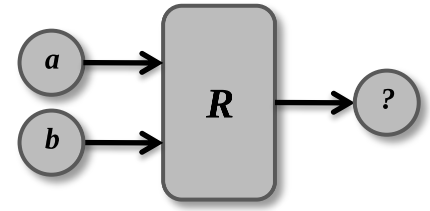
A Commercial Message From Scope

When we name a value, like x, we have to be careful when we reuse that name for something else, somewhere else. More precisely, we have to define the ***scope*** of x. The scope of x is its reach, its persistence, how long and in what context it will retain both its value and its ***meaning***. Scoping concerns have challenged many a computer scientist.

Some 1-input (unary) operations are invertible, some, like square root are not. Binary (2-input) operations are not invertible when we cannot recover which pair of inputs produced the output value. One might ask if an algebra of unary operations, by retaining invertibility, would retain important properties, like decidability, introduced by Gödel. For example, one could replace the binary operation of addition, with an equivalent succession of unary operations that add 1 to the input.

When we name functions, just as when we name variables, issues of scope arise, and the same considerations apply.

SYMMETRIC, REFLEXIVE, TRANSITIVE



A relation R is ***symmetric*** if R(a,b) and R(b,a) are both true.

Consider the relation, ***“sits on”***. If I sit on you, you cannot sit on me and vica versa. Therefore the relation “sits on” is not symmetric however awkward that might be. Now consider the relation, **“in same room”**. If I am in the same room as you, you are in the same room as me. Therefore, the relation, “in same room” is symmetric. Try this with various prepositional phrases to amuse yourself. It is extremely informative.

A relation R is ***reflexive*** if it is true for itself, that is if R(a, a) is true.

Consider the relation “in same room” again. If I am in a room, then I am in the same room as myself. This may sound like a silly tautology, but it is quite useful.

A relation R is ***transitive*** if R(a ,b) and R(b, c) imply that R(a, c)

If Bob is in the same room as Bill and Bill is in the same room as Jill then Bob is in the same room as Jill. Therefore “in same room” is transitive.

A Relation that is Symmetric, Reflexive and Transitive is called an ***Equivalence Relation*** and has the symbol ***‘=‘*** in mathematics and ***‘==‘*** in some programming languages.

UNIQUENESS

Uniqueness is a mathematical buzz word meaning, "there is only one solution". If an equation has more than one solution then it is not unique. In the pre-computer era of the 20th century scientists and engineers prized uniqueness because it meant there was only one solution and once you found it, you were done. Not only were you done, you didn't have to spend time looking for other possible solutions.

In general, linear systems of equations, like the ones described in the last lecture have the property that their solutions, if they exist are unique.

The phrase, "one and only one" when applied to the solution of an equation implies uniqueness. The proper notation for this assertion uses a funny backwards E for existence followed by an exclamation point to indicate uniqueness. When there is only one solution $\boldsymbol{k}$, we say, "there exists one and only one solution" and use the symbol, $\exists$**.**

To denote membership in a set we use a symbol that stands for "element of" or $\in$.
To denote a set like the natural numbers used for counting {0, 1, 2, 3…} we use the symbol $\mathbb{N}$
Putting these all together we have:

$$\exists\, \boldsymbol{k} \in \mathbb{N}: \boldsymbol{k} + 2 = 4$$

Which means, there exists one and only one $\boldsymbol{k}$ in the natural numbers such that $\boldsymbol{k} + \boldsymbol{2} = \boldsymbol{4}$.
That's uniqueness in a nutshell.

Closure

The notion of closure is this:

Given an operator:

for example addition or subtraction, and

Given a set of numbers:

for example the counting numbers, {1, 2, 3 … }

Closure:

$$\sqrt{-1}$$

The square root of a negative real isn't.

Is it possible to generate any numbers that are not members of the original set?

Addition is closed, because if you add any two counting numbers a, and b, you always get a counting number as a result (output).

Subtraction violates closure within counting numbers, since it is possible to subtract two counting numbers and not get a counting number. For example 2 – 3 is not a counting number.

In the Geometry Expressions™ examples above, when the test point moved left of zero, a square root of a negative x is attempted. The curve disappears, because closure has been violated.

Progress in mathematics has been a succession of closure violations and repairs.
Complex numbers are an example of this. More on that in subsequent chapters.

End