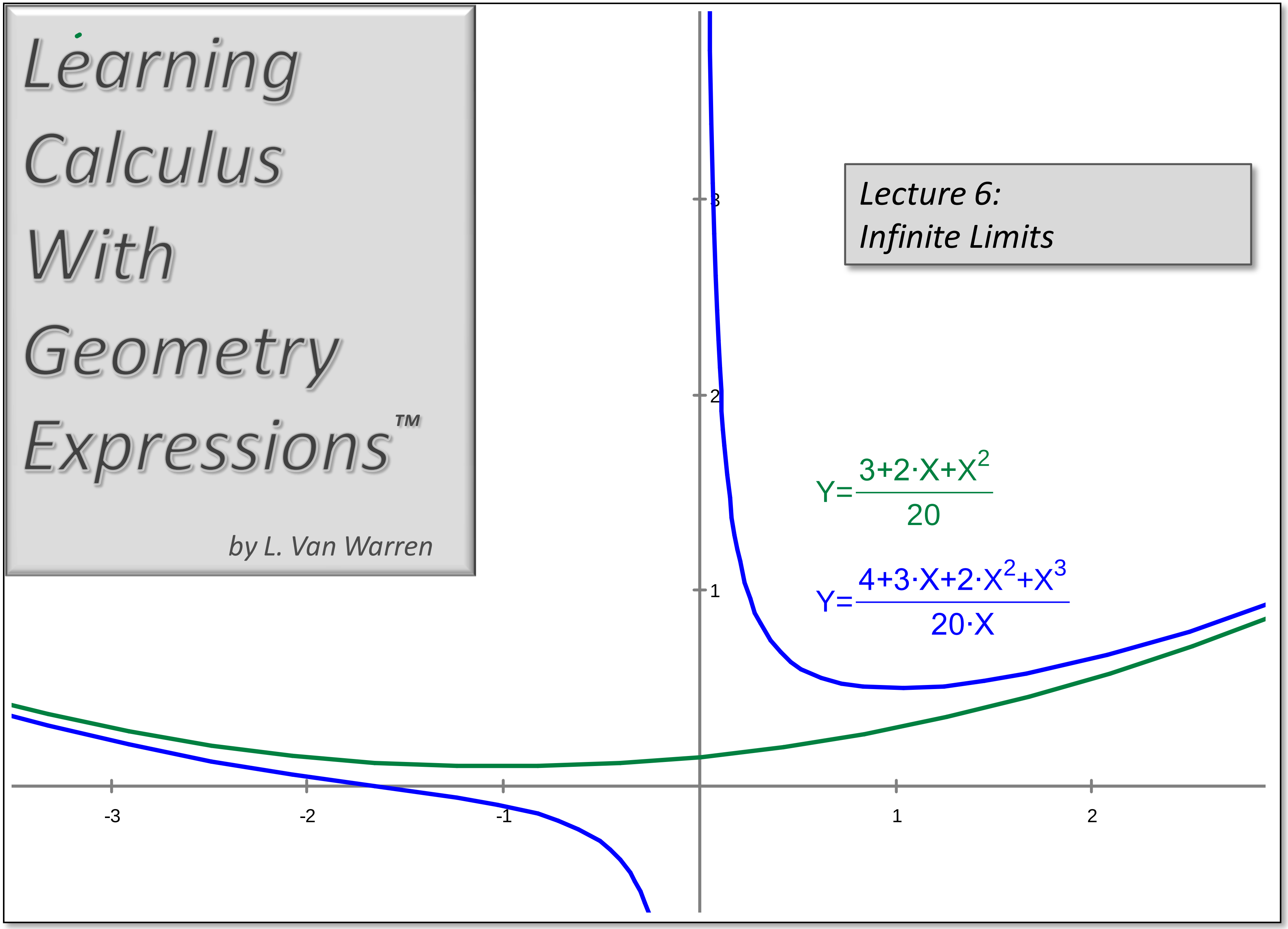
Learning Calculus With Geometry Expressions™

by L. Van Warren

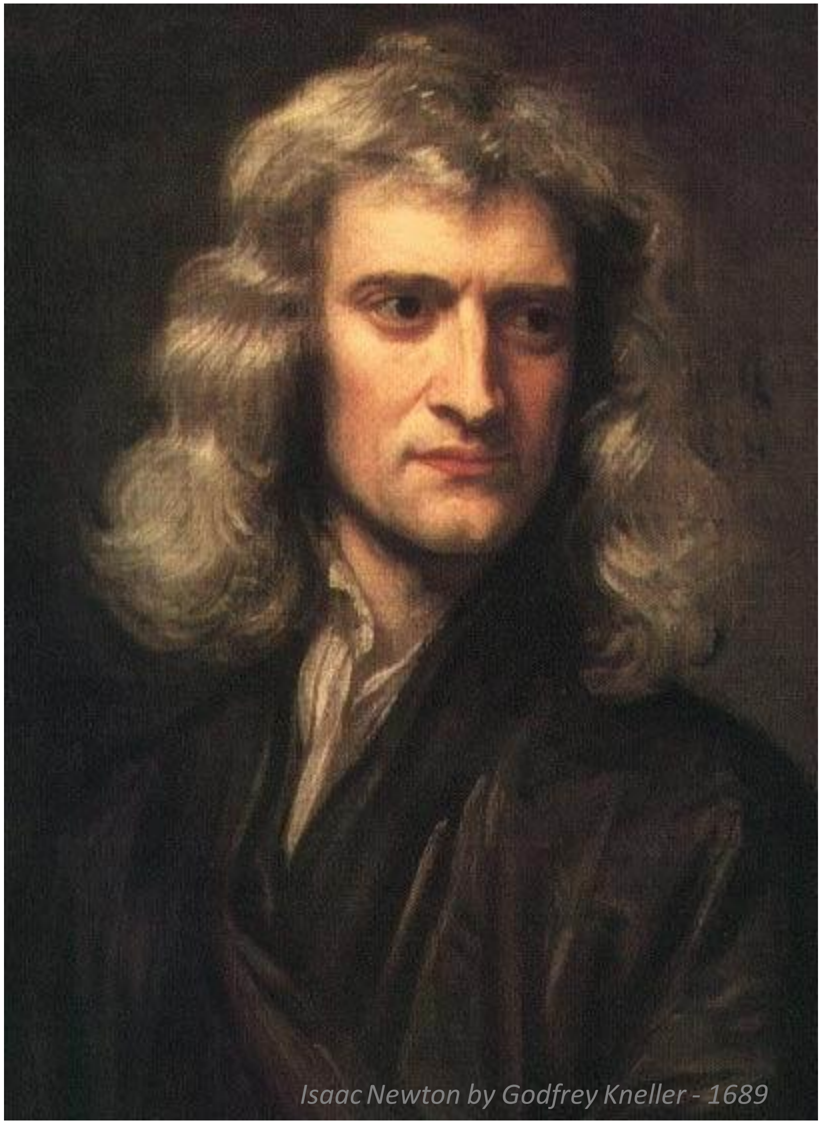
Lecture 6: Infinite Limits



Chapter 2: Limits

Lecture	Topic
5	Finite Limits
6	**Infinite Limits**
7	Continuity
8	Discontinuity
9	Precise Definition of The Limit

Inspiration



Isaac Newton

Three Laws That Would Change the World.

Besides inventing calculus, making advances in optics, cracking the rainbow and numerous other contributions, Isaac Newton discovered three laws of motion that would forever alter man's ability to understand, predict and control.

These three laws of motion are:

1) Objects move in a straight line unless forced otherwise.

2) $F = m\dfrac{d^2x}{dt^2}$

3) For every action there is an equal and opposite reaction.

Einstein and others would later perfect these laws for near-light speeds, but this was the beginning…

Infinite Limits

Consider the Limit whose name is ***L*** below:

$$L = \lim_{x \to 0} \frac{1}{x^2}$$

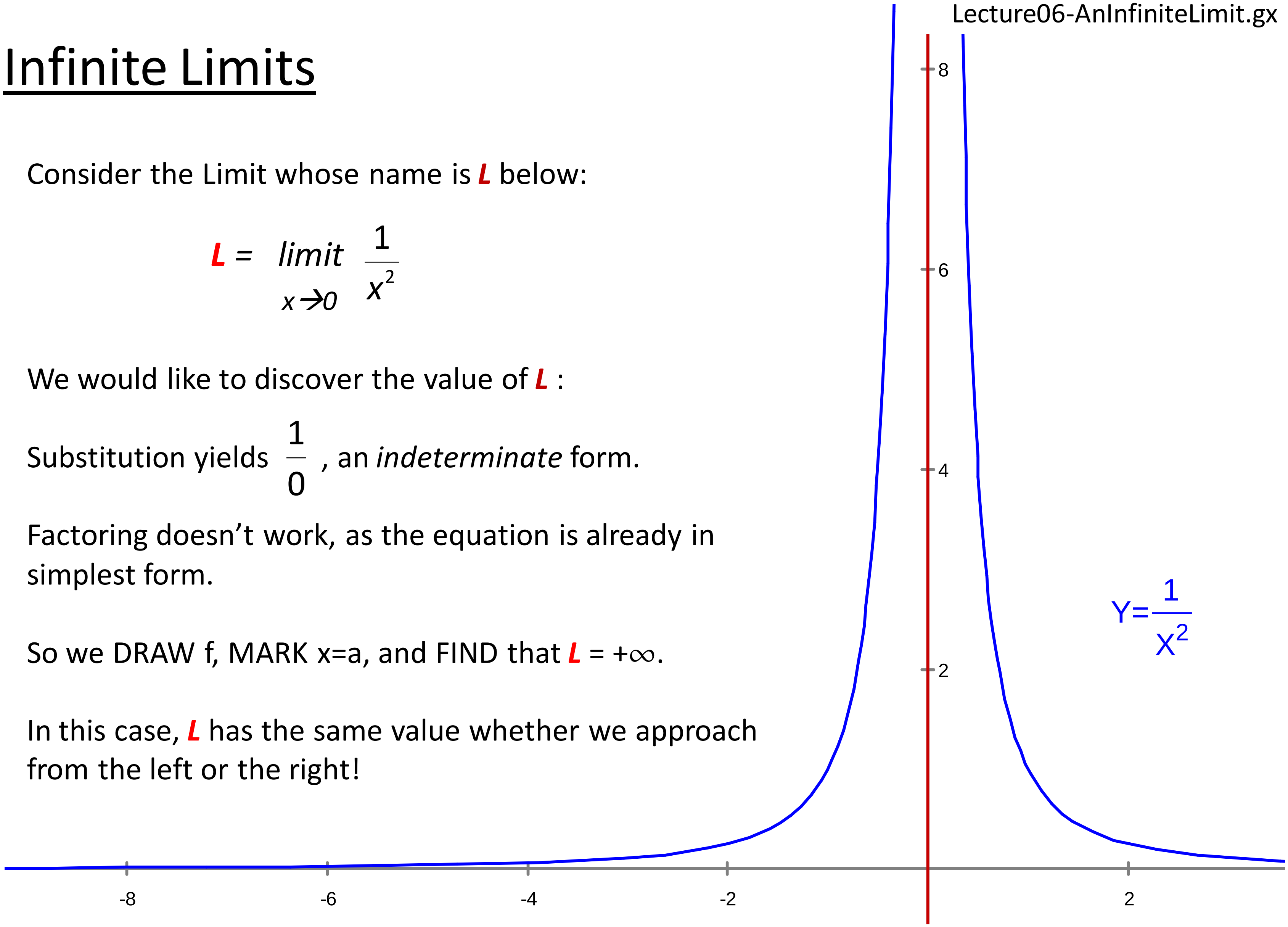
We would like to discover the value of ***L*** :

Substitution yields $\frac{1}{0}$, an *indeterminate* form.

Factoring doesn't work, as the equation is already in simplest form.

So we DRAW f, MARK x=a, and FIND that ***L*** = +∞.

In this case, ***L*** has the same value whether we approach from the left or the right!



Infinite Limits

The limiting ***y***-value of the function can be infinite as shown in the example on the right.

The ***x***-value can also be allowed to approach infinity.

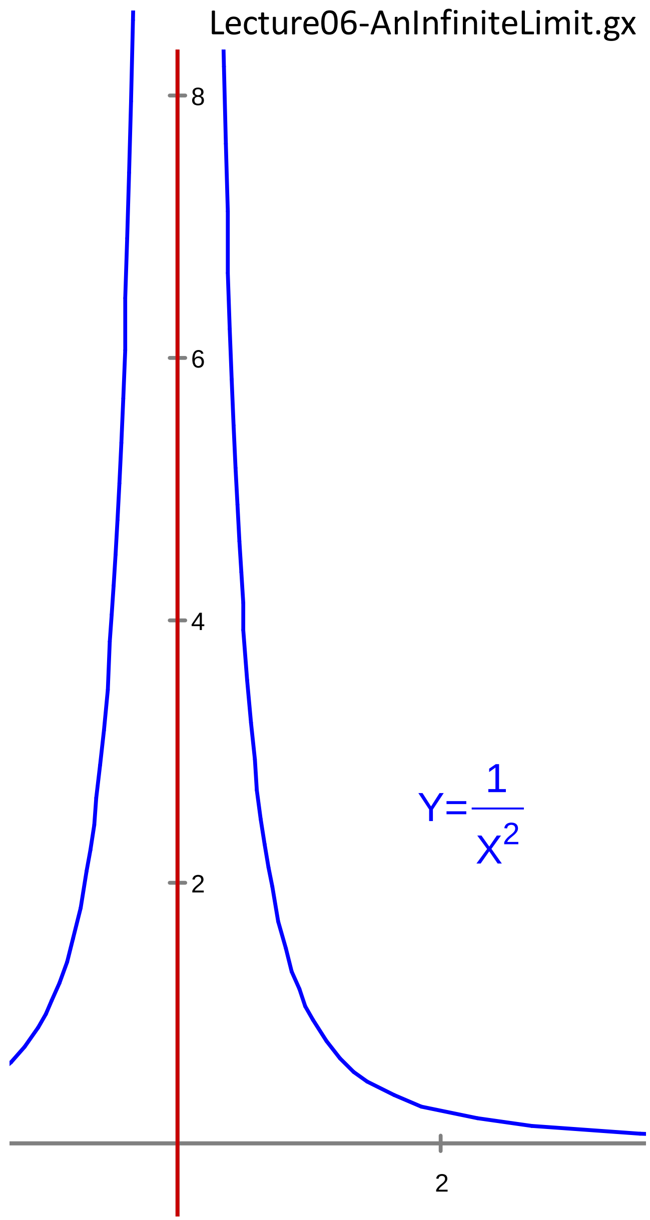
For example the limit L defined as:

$$L = \lim_{x \to \infty} \frac{1}{x}$$

exists and has a finite value.

EXERCISE

Use the DRAW, MARK, FIND method to discover the value of ***L*** for the case above. Draw the result using Geometry Expressions™.



One-Sided Limits At x = 0

When we take the limit, we think of x as starting at some arbitrary value and approaching the limiting value.

If we approach the limit as x → 0 from values less than zero, we write:

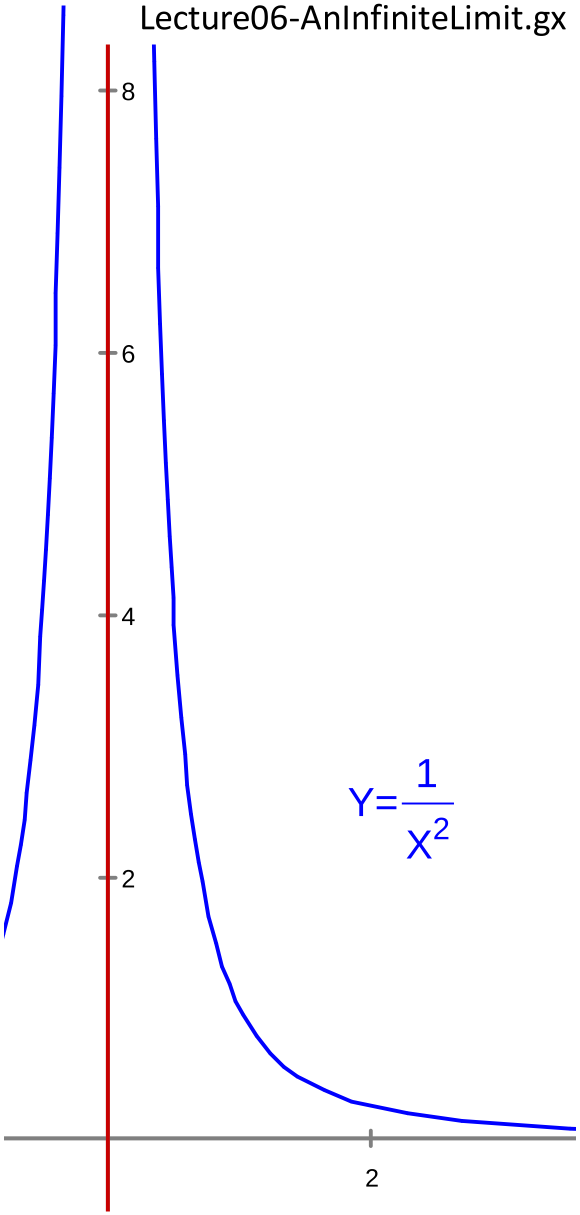
$$\lim_{x \to 0^-} f(x) = f(0)$$

and we say, "The limit of f(x) as ***x*** goes to ***0*** from the left."

If we approach the limit as x → 0 from values greater than zero, we write:

$$\lim_{x \to 0^+} f(x) = f(0)$$

and we say, "The limit of f(x) as ***x*** goes to ***0*** from the right."



One-Sided Limits at x = a

Lecture06-AnInfiniteLimitAtA.gx

When we take the limit, we think of x as starting at some arbitrary value and approaching the limiting value.

If we approach the limit as x → a from the left, we write:

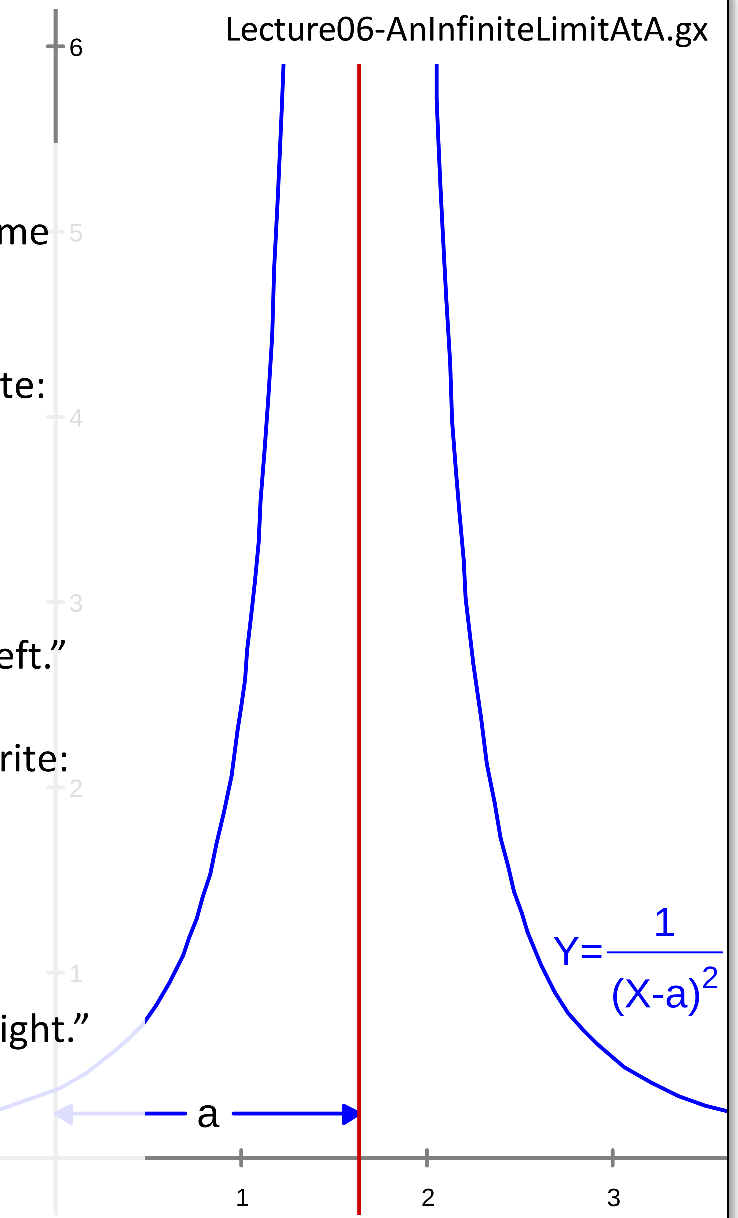
$$\lim_{x \to a^-} f(x) = f(a)$$

and we say, "The limit of f(x) as ***x*** goes to ***a*** from the left."

If we approach the limit as x → a from the right, we write:

$$\lim_{x \to a^+} f(x) = f(a)$$

and we say, "The limit of f(x) as ***x*** goes to ***a*** from the right."



Asymptotes

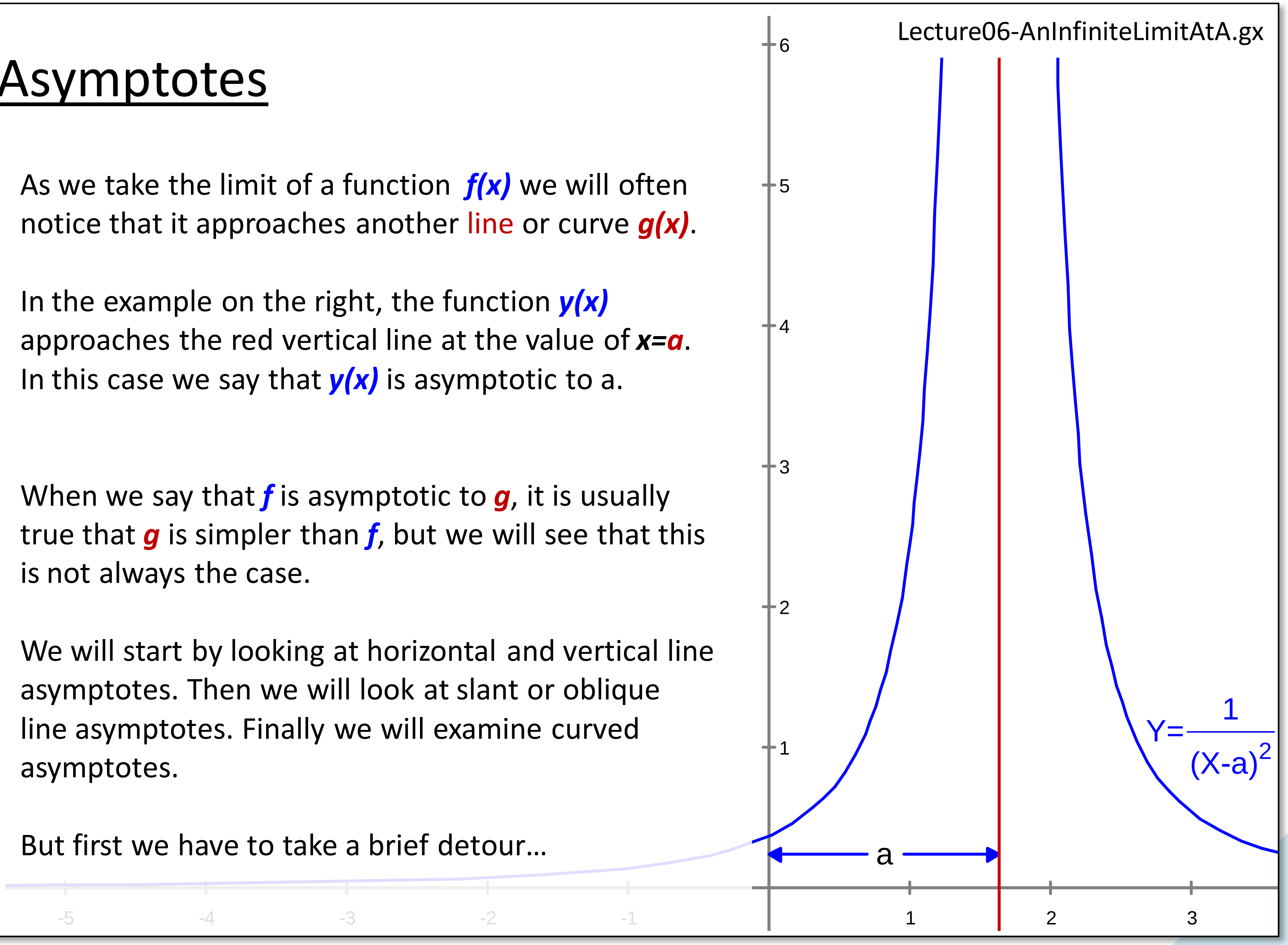
As we take the limit of a function ***f(x)*** we will often notice that it approaches another line or curve ***g(x)***.

In the example on the right, the function ***y(x)*** approaches the red vertical line at the value of ***x=a***. In this case we say that ***y(x)*** is asymptotic to a.

When we say that ***f*** is asymptotic to ***g***, it is usually true that ***g*** is simpler than ***f***, but we will see that this is not always the case.

We will start by looking at horizontal and vertical line asymptotes. Then we will look at slant or oblique line asymptotes. Finally we will examine curved asymptotes.

But first we have to take a brief detour…



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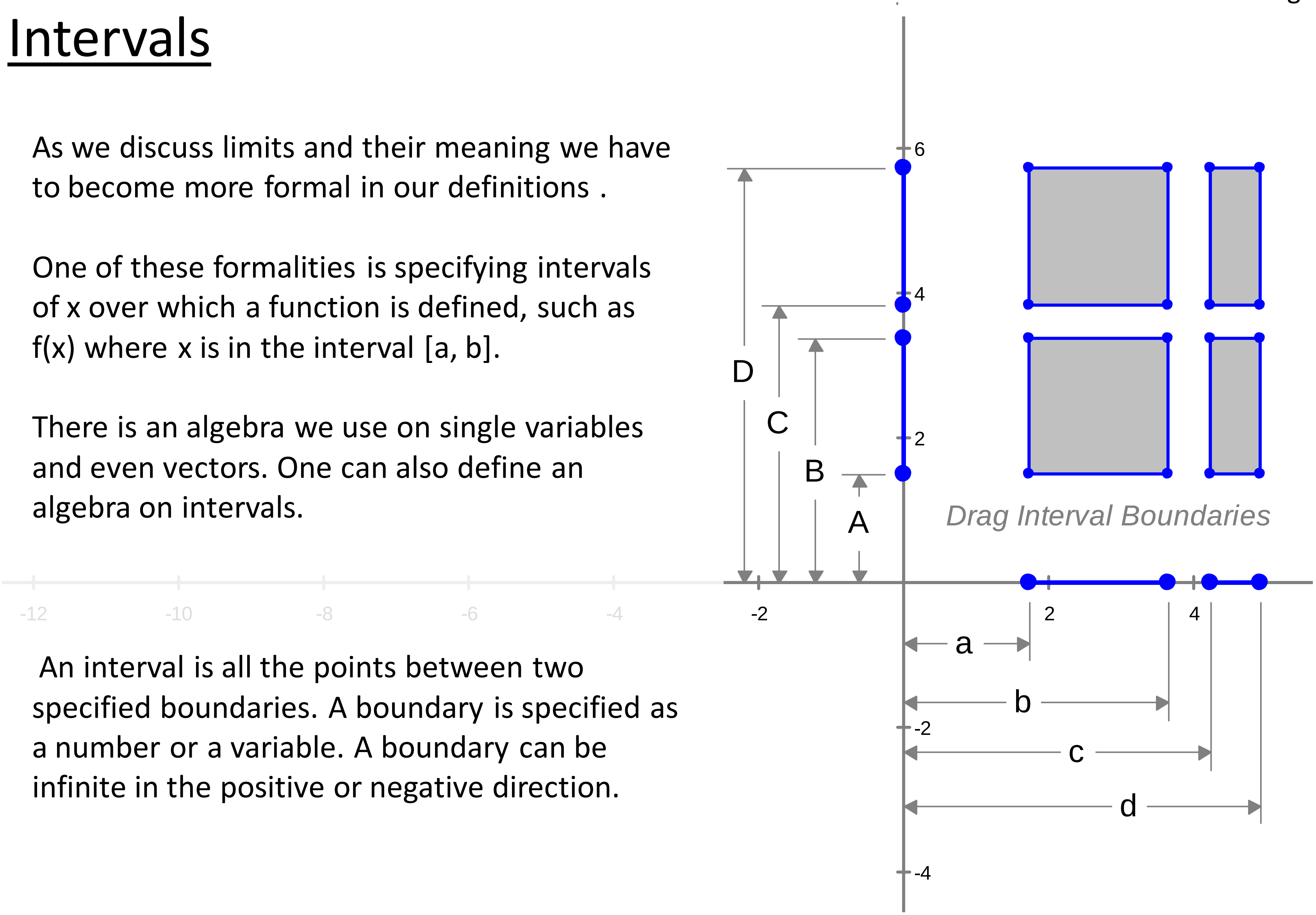
Intervals

As we discuss limits and their meaning we have to become more formal in our definitions .

One of these formalities is specifying intervals of x over which a function is defined, such as f(x) where x is in the interval [a, b].

There is an algebra we use on single variables and even vectors. One can also define an algebra on intervals.



An interval is all the points between two specified boundaries. A boundary is specified as a number or a variable. A boundary can be infinite in the positive or negative direction.

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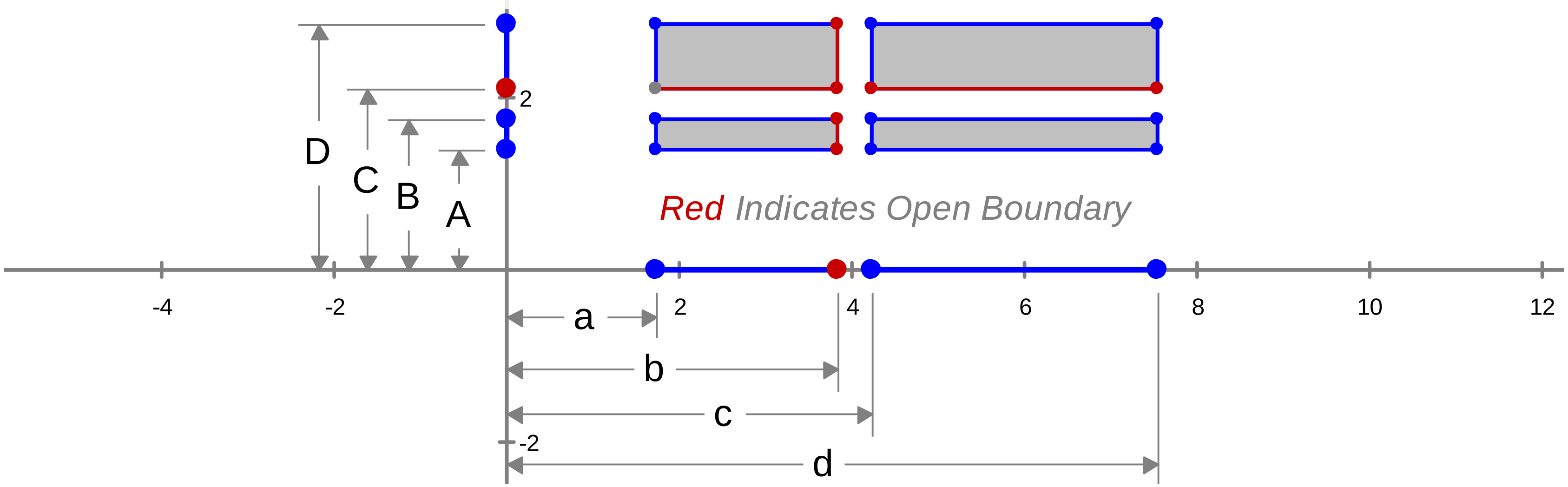
Interval Algebra

One can specify an interval on the real valued line by the notation:

[a, b] if the interval is closed and includes the end values a and b.
(a,b) if the interval is open and does not include the end values a and b.
[a,b) if the interval is closed on the left and open on the right with respect to a and b.
(a,b] if the interval is open on the left and closed on the right with respect to a and b.

The numbers a and b that constitute the boundaries of the interval can be finite or infinite. When a boundary is infinite, the interval must be open on that side, since there are different sizes of infinity.



Expressions in Interval Algebra

One can give an interval a name, like iAB= [2,3)

One can combine intervals using the Boolean operators of AND, OR, & NOT.

Boolean combinations of intervals are not closed with respect to intervals themselves:

For example NOT (2,3] = (-∞, 2] AND (3, +∞)

However they can be simplified to the simplest possible Boolean expression.

Some Boolean expressions will collapse to an interval.

For example NOT((-∞, 2] AND (3, +∞)) = (2,3]

Expressions that don't can be evaluated piecewise, with the Boolean relationship retained.

Intervals are useful for fuzzy logic, numerical ideas and representing uncertainty.

Horizontal Asymptotes

Lecture06-HorizontalAsymptotes.gx

Having concluded our detour on interval arithmetic we can now present the definition of a horizontal asymptote.

*Given **f(x)** on the open interval **(a, ∞)**:*

L *= limit **f(x)*** *means*
x→∞

f(x) *approaches **L** as x approaches ∞.*

Said another way:

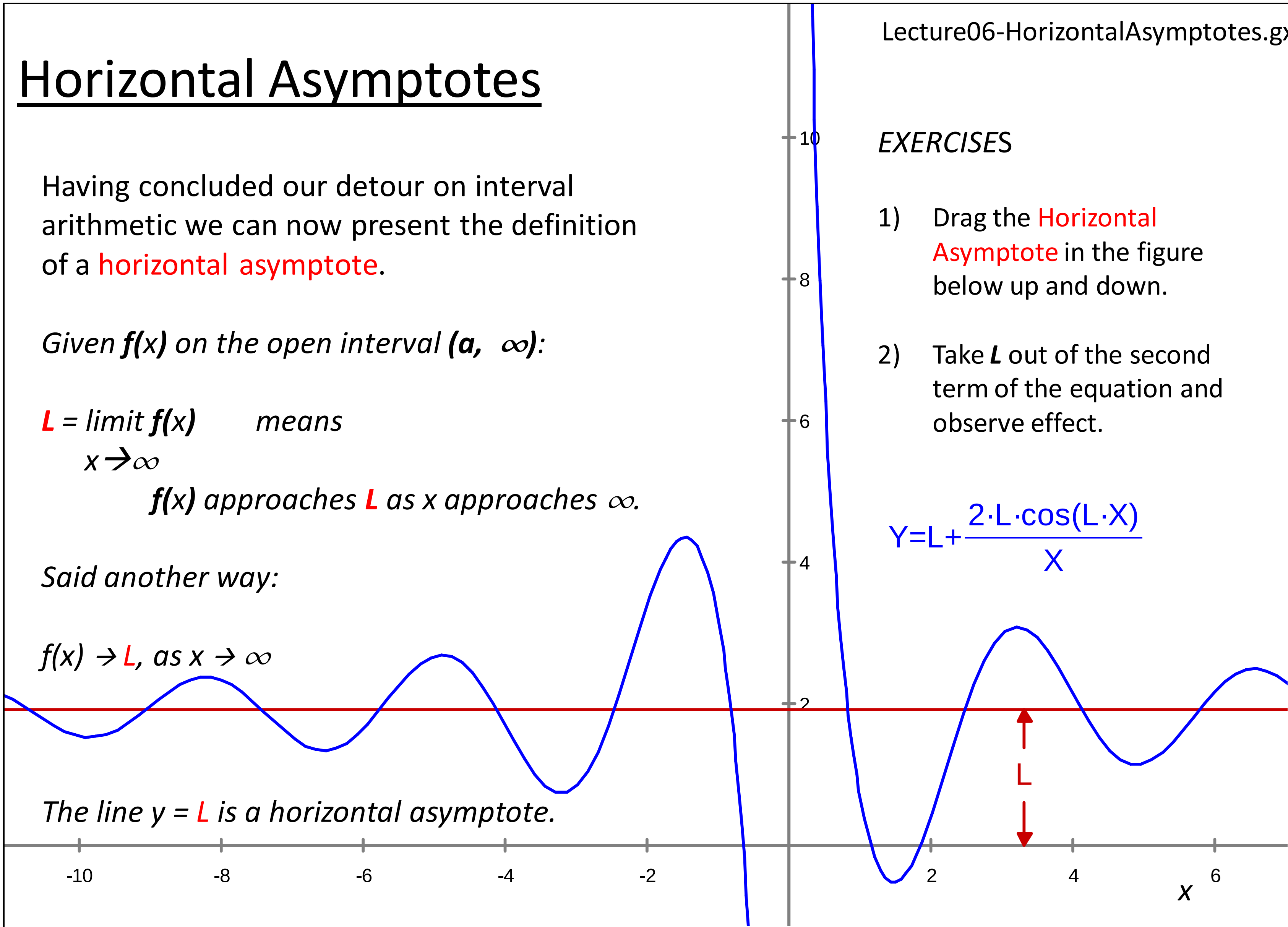
f(x) → L, as x → ∞

The line y = L is a horizontal asymptote.

EXERCISES

1) Drag the Horizontal Asymptote in the figure below up and down.

2) Take ***L*** out of the second term of the equation and observe effect.



Lecture06-HorizontalAsymptotes.gx

Horizontal Asymptotes

A second definition of a horizontal asymptote.

Given ***f(x)*** *on the open interval* ***(-∞, a)****:*

L *= limit* ***f(x)*** *means*
x → -∞
f(x) *approaches* ***L*** *as x approaches -∞.*

Said another way:

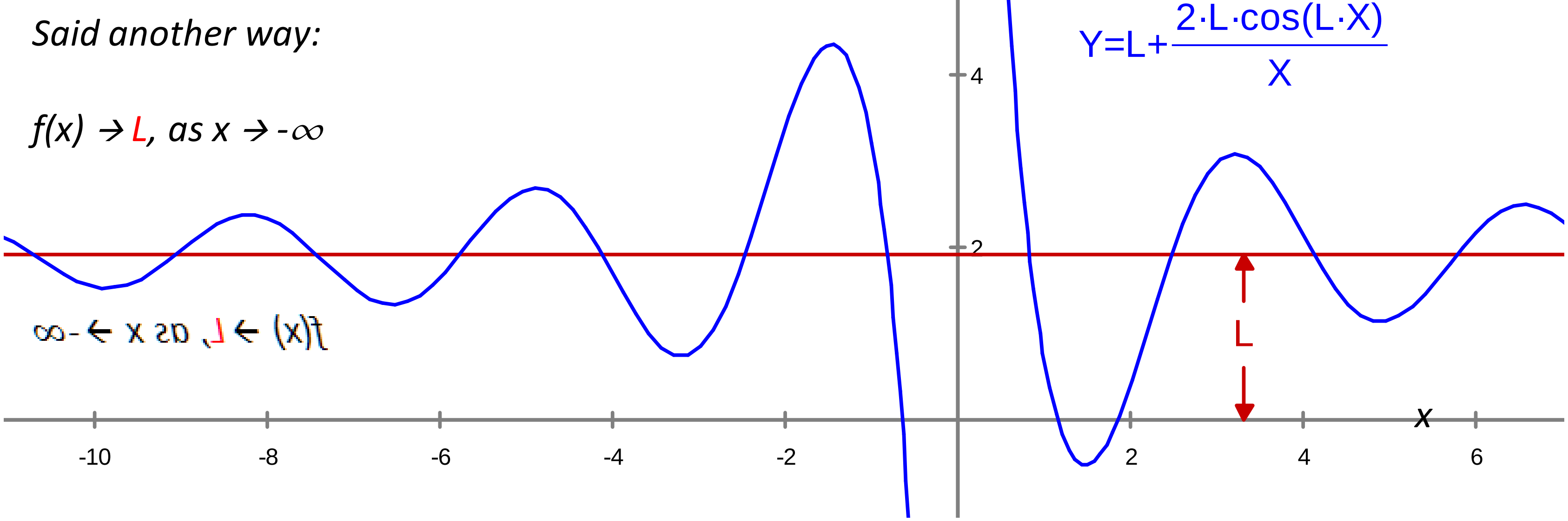
f(x) → L, as x → -∞

f(x) → L, as x → -∞

The astute observer notices that the previous page, in taking the limit to +∞, had neglected the case of taking the limit to -∞.

This page provides the symmetric definition, creating a pair of limits, first to the right, and this, to the left.

$$Y=L+\frac{2\cdot L\cdot \cos(L\cdot X)}{X}$$



Lecture06-VerticalAsymptote.gx

Vertical Asymptotes

We can also define the notion of a vertical asymptote.

In this case we use the one-sided limit.

The one-sided limit is written:

$$L = \lim_{x \to 0^+} f(x)$$

and we say:

L equals the limit as x goes to zero from the right, never actually reaching zero.

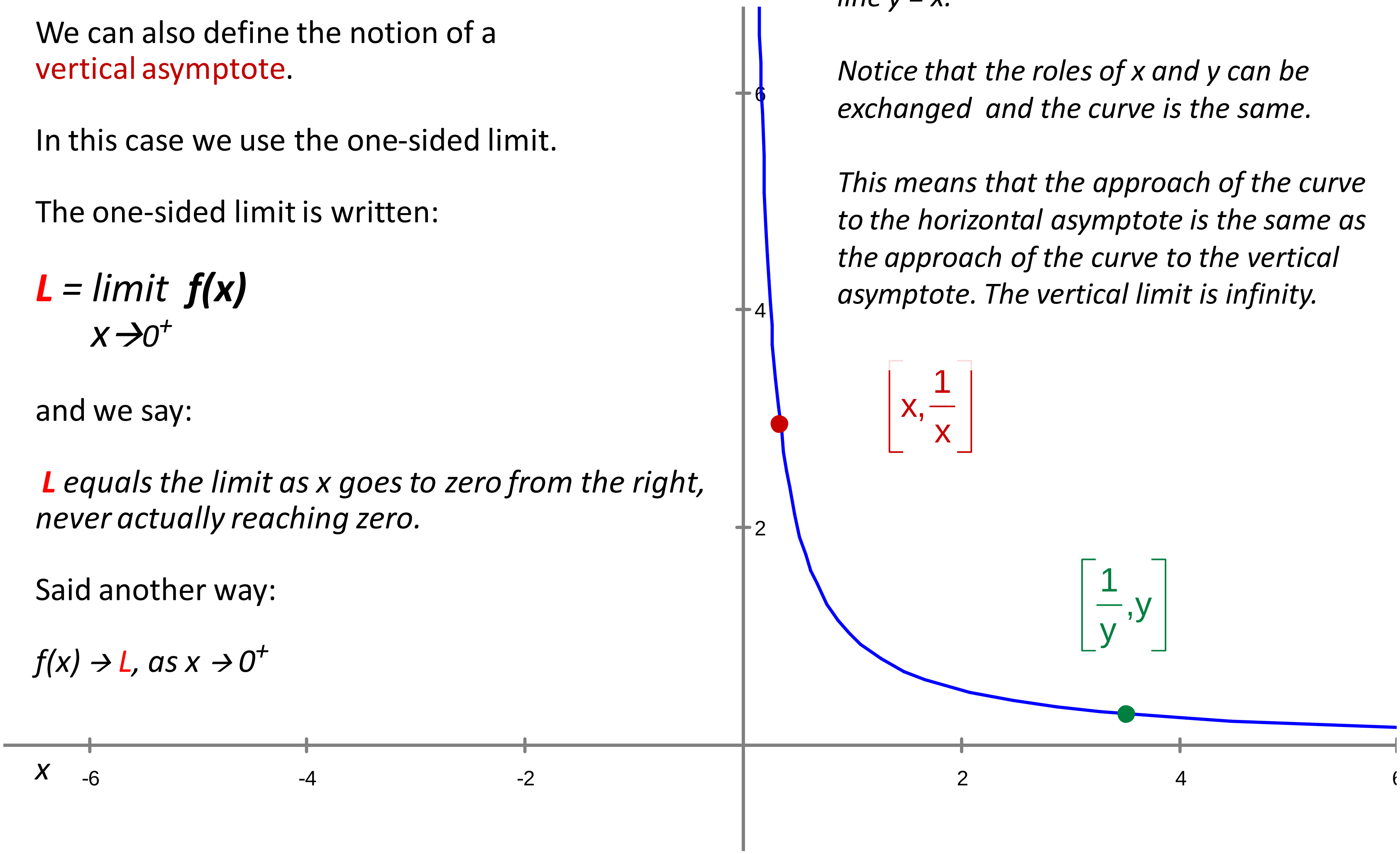
Said another way:

$f(x) \to L$, as $x \to 0^+$

The curve y = 1/x is symmetric about the line y = x.

Notice that the roles of x and y can be exchanged and the curve is the same.

This means that the approach of the curve to the horizontal asymptote is the same as the approach of the curve to the vertical asymptote. The vertical limit is infinity.



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Slant Asymptotes

Whereas we just took the limit of the *function itself* in the horizontal asymptote case, in the slant case we take the limit of the function minus its slant asymptote.

For this example the slant asymptote is written:

$$L = \lim_{x \to \infty} \left((x + 1/x) - (x) \right) = 0$$

and we say:

L *equals the limit as x goes to positive infinity of the function (x + 1/x) minus the function of its slant asymptote, g(x) and in this case the limit is zero.*

Said another way in shorthand:

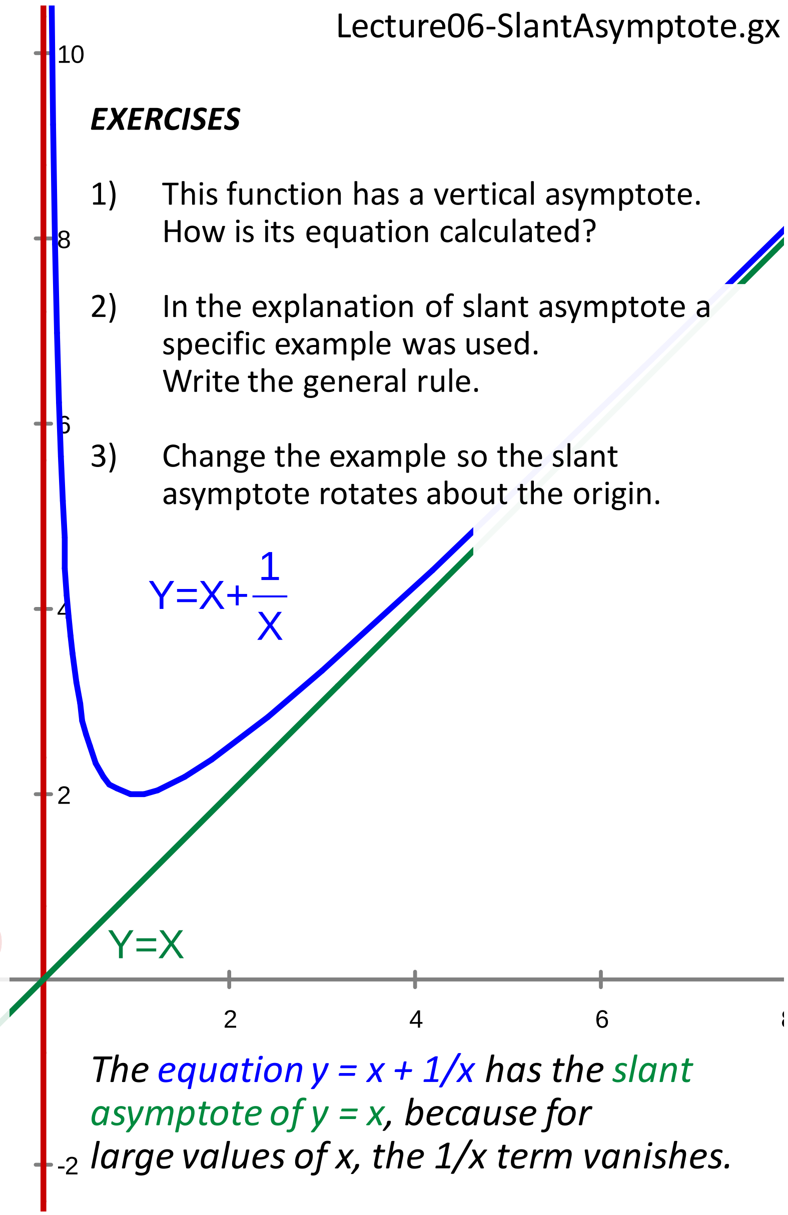
f(x) - g(x) $\to L \to 0$*, as x* $\to \infty$

EXERCISES

1) This function has a vertical asymptote. How is its equation calculated?

2) In the explanation of slant asymptote a specific example was used. Write the general rule.

3) Change the example so the slant asymptote rotates about the origin.



The equation y = x + 1/x has the slant asymptote of y = x, because for large values of x, the 1/x term vanishes.

Lecture06-CurvedAsymptote.gx

Curved Asymptotes

In our previous example of slant asymptotes we noticed how one function could become asymptotic to another in the limit. There is no reason why an asymptote has to be a straight line, it can in fact be a curve itself.

Our shorthand gave a hint that this was true:

$f(x) - g(x) \rightarrow L \rightarrow 0, \text{ as } x \rightarrow \infty$

We merely generalize g(x) to include curves as well as lines.

Consider the right-handed limit written:

$$\mathbf{L} = \lim_{x \to \infty} \left(f(x) - g(x) \right) = 0$$

We say:

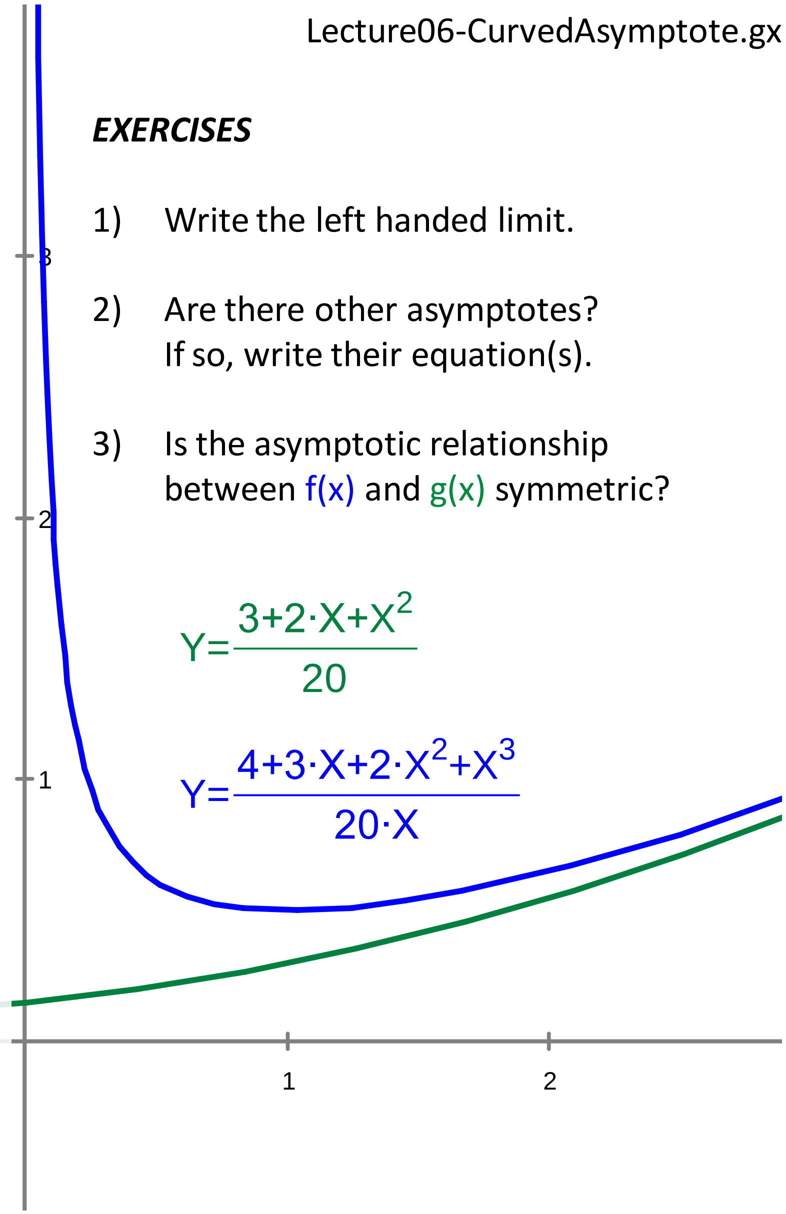
***L** equals the limit as x goes to positive infinity of the function f(x) minus the function of its curved asymptote g(x) and in this case the limit is zero.*

EXERCISES

1) Write the left handed limit.

2) Are there other asymptotes? If so, write their equation(s).

3) Is the asymptotic relationship between f(x) and g(x) symmetric?



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A Limit Theorem

We have worked in shorthand and in plain language. Now we will express our ideas as pure symbolic expressions:

If a/b > 0 then:

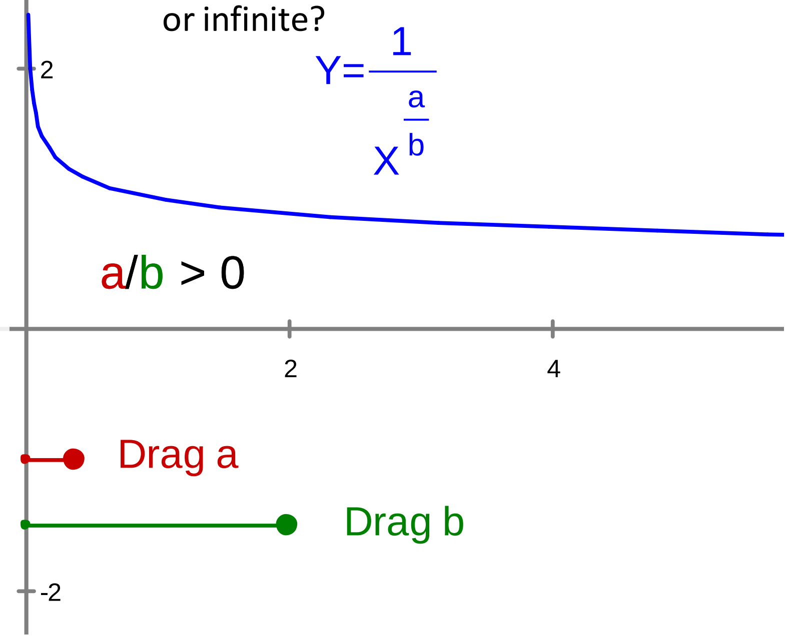
$$L = \lim_{x \to \infty} \left\{ \frac{1}{x^{\frac{a}{b}}} \right\} = 0$$

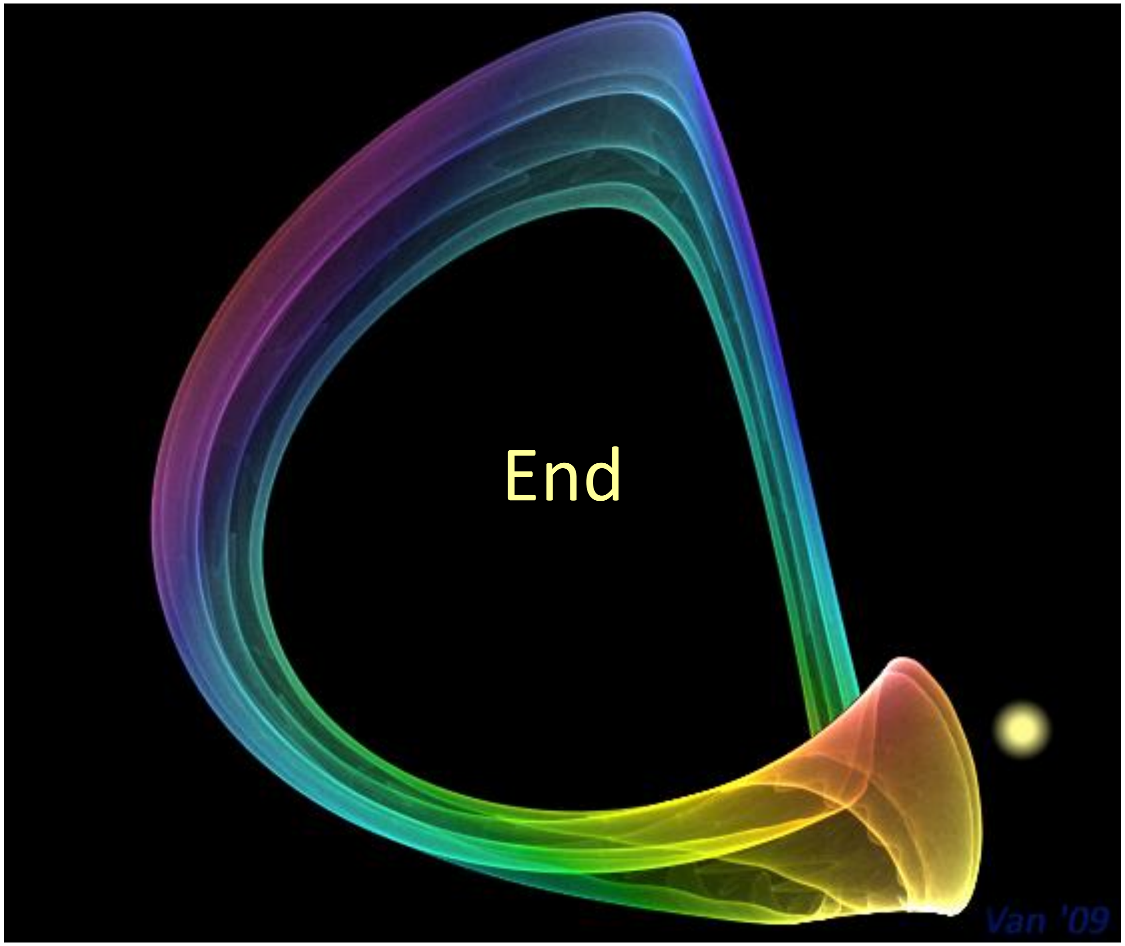
If a/b > 0 AND $x^{\frac{a}{b}}$ is defined for all x, then:

$$L = \lim_{x \to -\infty} \left\{ \frac{1}{x^{\frac{a}{b}}} \right\} = 0$$

EXERCISES

1) Drag a and b in both positive and negative directions..
2) When does the function appear to go to zero in the limit.
3) Can you make the limit infinite?
4) Are there limits that are neither zero or infinite?

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