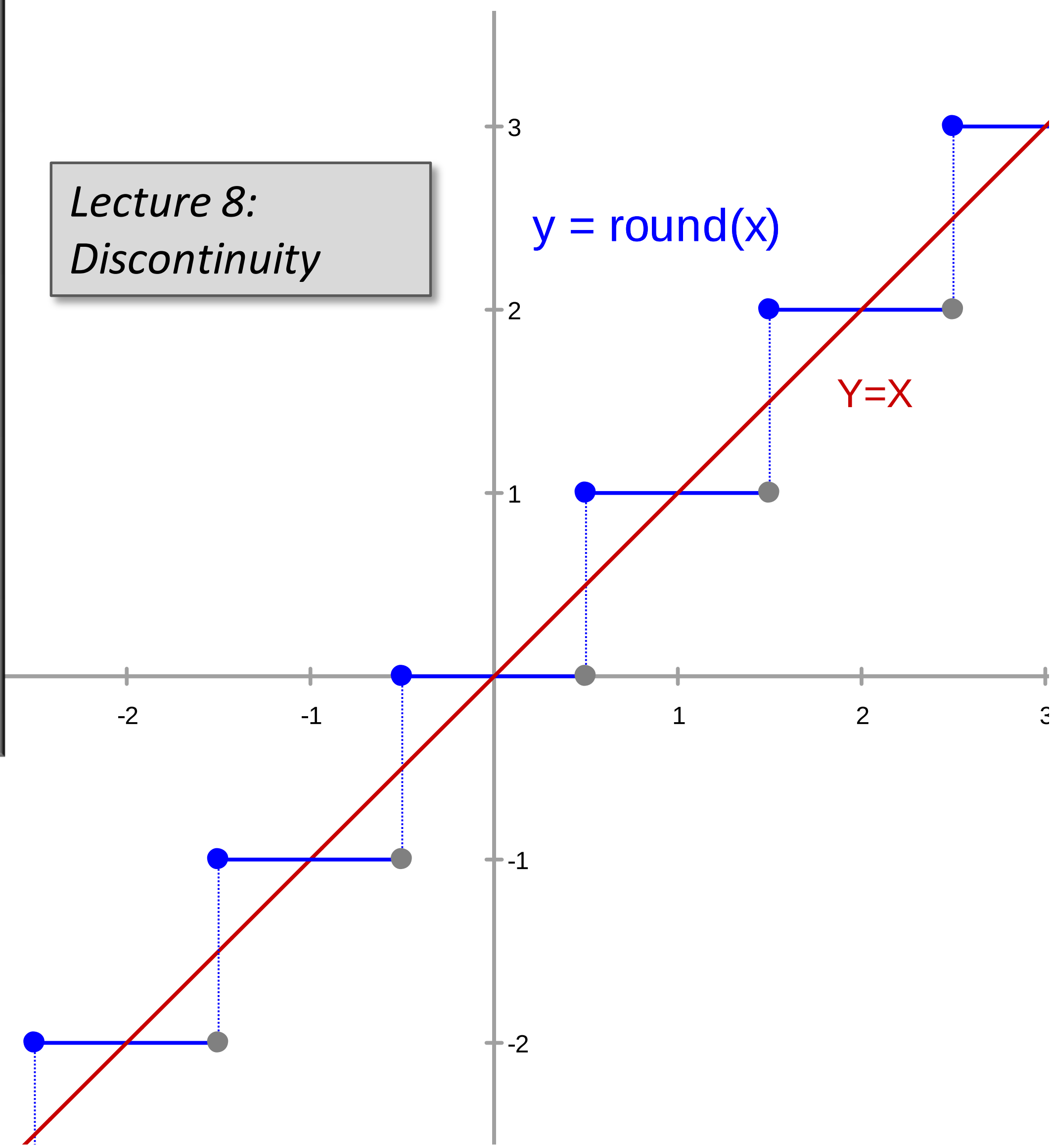
Learning Calculus With Geometry Expressions™

by L. Van Warren

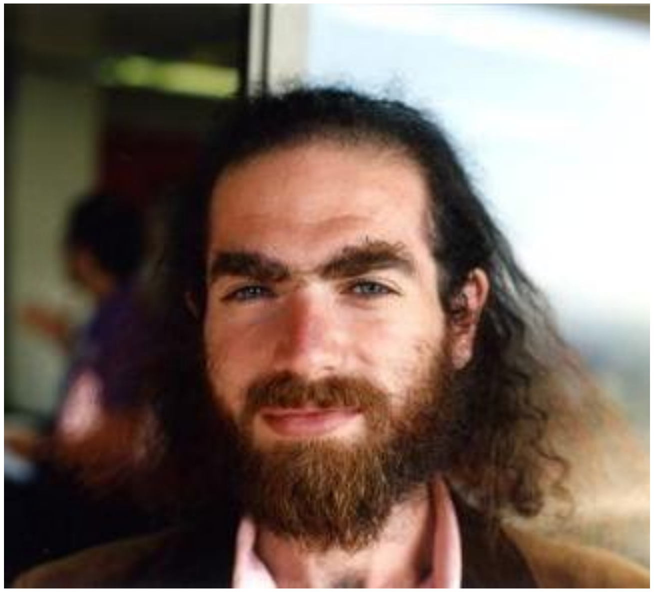
Lecture 8: Discontinuity



Chapter 2: Limits

Lecture	Topic
5	Finite Limits
6	Infinite Limits
7	Continuity
8	**Discontinuity**
9	Precise Definition of The Limit

Inspiration



- Bergman / Mathematisches Institut Oberwolfach

Grigori Perelman
is a Russian mathematician who distinguished himself by solving *Poincare's conjecture*, a Millennium Clay Price math problem with a million dollar reward.

He solved this problem in a novel way by modifying Richard Hamilton's program that models curvature as the flow of heat through a topological space. He corrected and extended the solution for exceptional cases that had stymied others.

He rejected the highest prizes in mathematics - the Field's medal and Clay Prize for his feat. After other mathematicians allegedly claimed his work as their own he resigned from mathematics saying, "Everybody understood that if the proof is correct then no other recognition is needed."

Dr. Perelman is a talented violinist and enjoys table tennis. He values honesty and integrity in mathematics beyond all else.

Discontinuity and Piecewise Functions

A function is discontinuous if it has gaps, jumps, or breaks. Piecewise functions are often discontinuous.

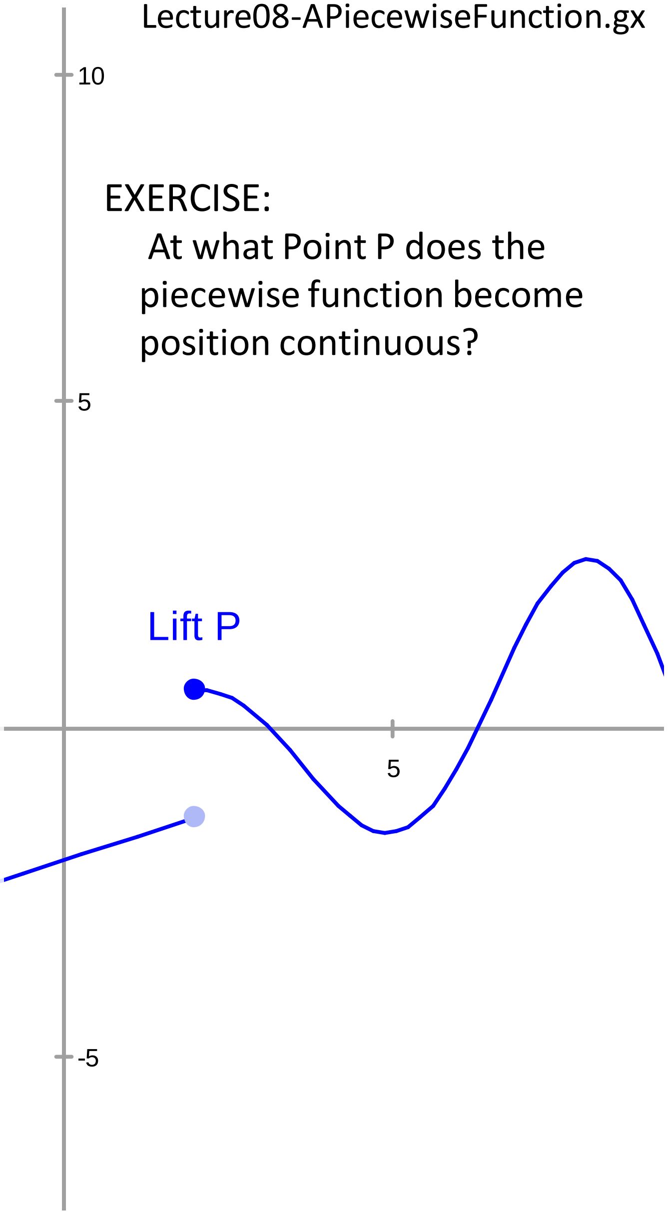
A piecewise function is defined for specific intervals, for example:

$$y = x/a - 2 \quad \text{for } x < 2 \text{ AND}$$
$$y = x \sin(x)/a \quad \text{for } x \geq 0$$

Words like IF, AND, OR, and NOT in the definition of a function are a tip off that you are dealing with a piecewise function. A function is position or C_0 continuous if you don't have to lift your pen from the paper while drawing. Even so if a function has a sudden peak or cusp, the slope may be discontinuous. We will show that one may use continuous techniques provided one remains in the interval over which the function is continuous.

Lecture08-APiecewiseFunction.gx

EXERCISE:
At what Point P does the piecewise function become position continuous?



Examples

Continuous

- polynomials

  $y = a + bx + cx^2 + \ldots$

- roots of n > 0

  $y = a\, x^{1/n}$

- sine functions

  $y = a \sin(x+\phi)$

Discontinuous

- floor

  $y = floor(x)$

- ceiling

  $y = ceiling(x)$

- round

  $y = round(x)$

Lecture08-FloorOfX.gx

Discontinuous Functions: floor(x)

This example shows the function floor(x) which maps real or integer input values into integers.

Symbolically we write:

$$floor(x) = largest\ n \leq x = max(n \leq x)$$

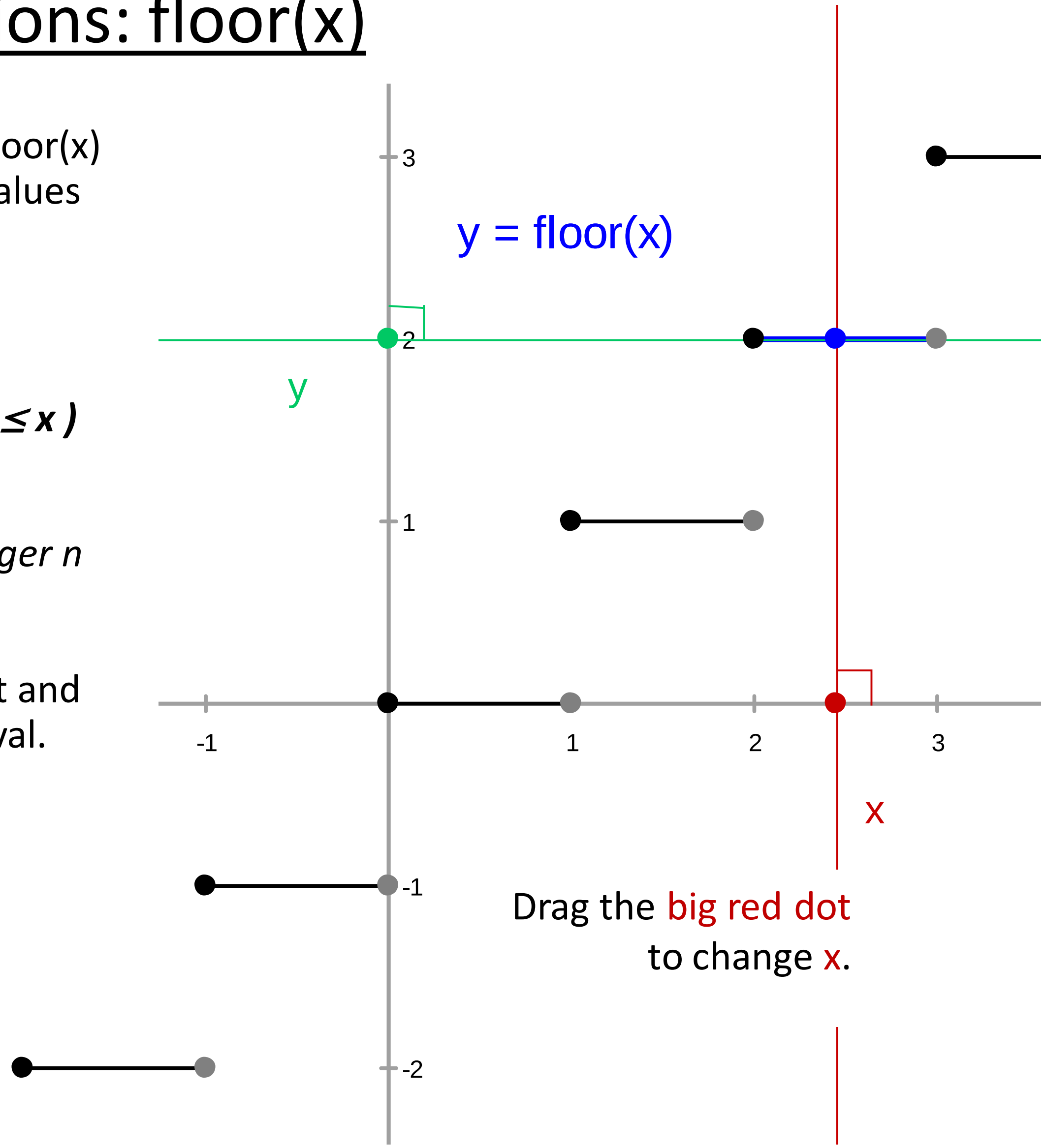
And say:

"The *floor of* x *is the largest integer n less than or equal to x.*"

Note that floor is closed on the left and open on the right side of the interval.

EXERCISE:

Prove ***floor(floor(x)) = floor(x) ;***



Discontinuous Functions: ceil(x)

This example shows the function ceil(x) which, like its cousin floor(x), maps real or integer input values into integers.

Symbolically we write:

ceil(x) = smallest n ≥ x = min(n ≥ x)

And say:

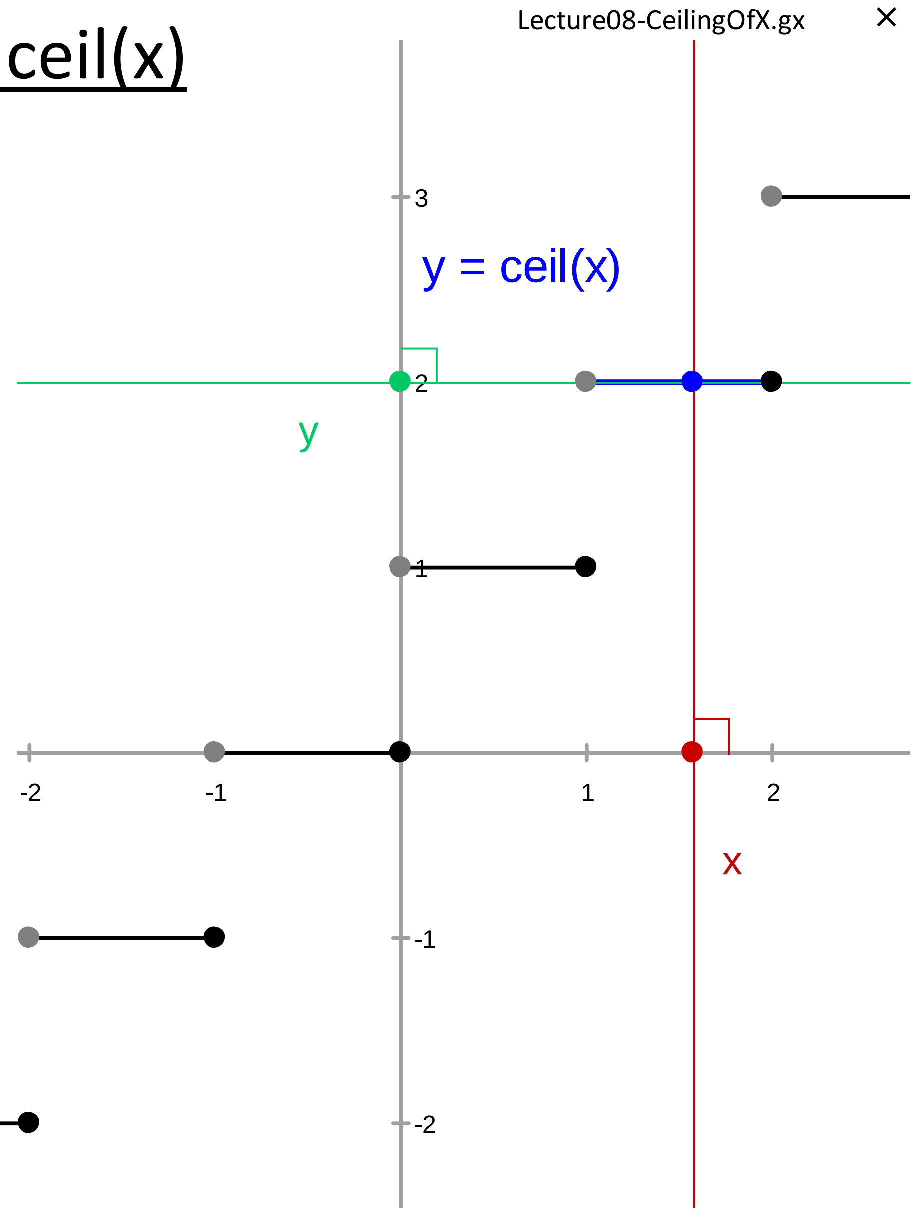
“Ceiling *of x is the smallest integer n greater than or equal to x.”*

Note that ceiling is open on the left and closed on the right side of the interval.

EXERCISE:

Prove ***ceil(x) = -floor(-x) ;***

Hint: Use View→Show All



Discontinuous Functions: round(x)

Lecture08-RoundX.gx

This example shows the function round(x) which, as its name implies, rounds a real or integer input value to the *nearest* integer.

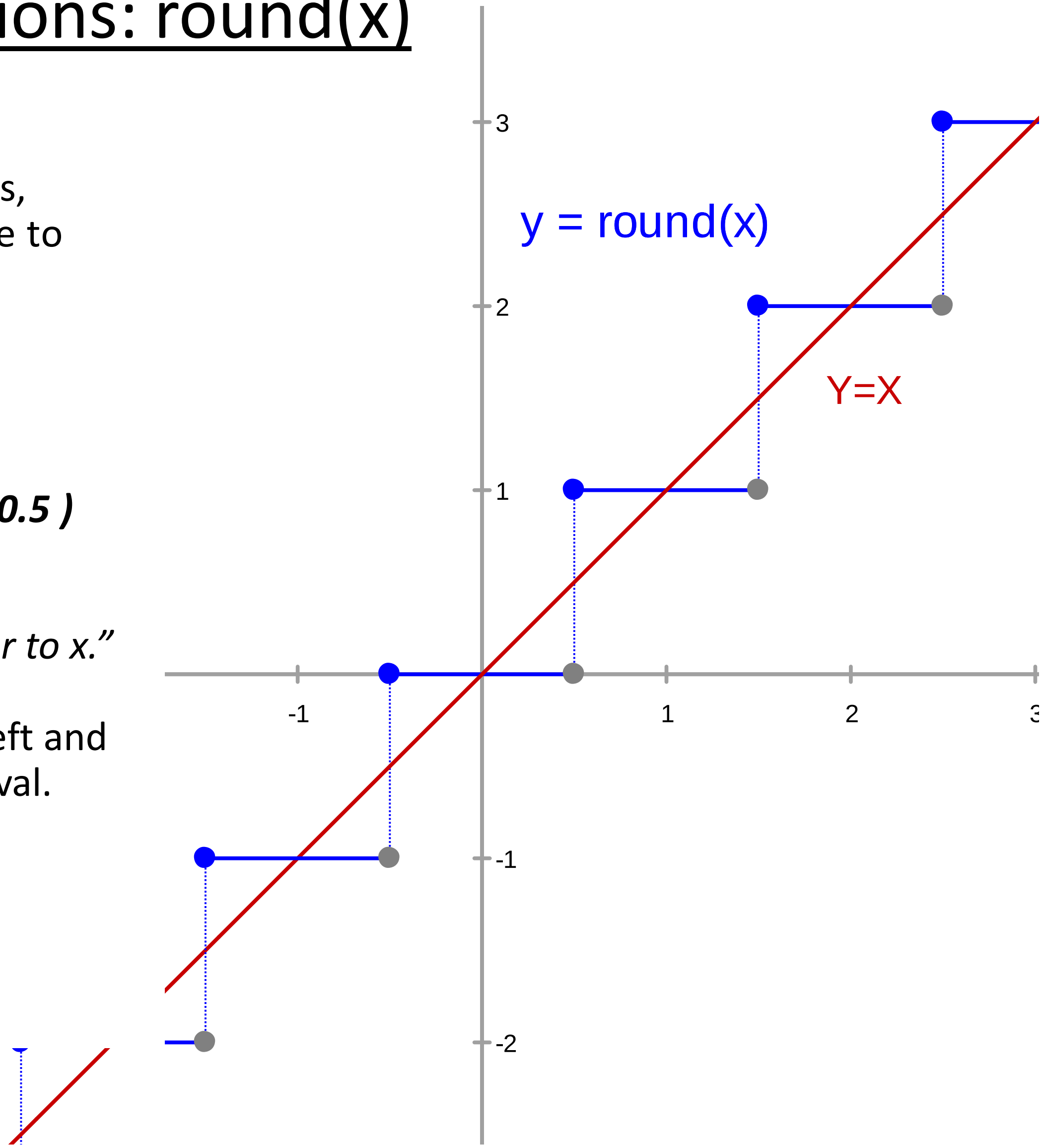
Symbolically we write:

round(x) =
largest n $\leq$ ***x+0.5 = max(n*** $\leq$ ***x+0.5)***

And say:
“round *of x is the nearest integer to x.”*

Note that round is closed on the left and open on the right side of the interval.

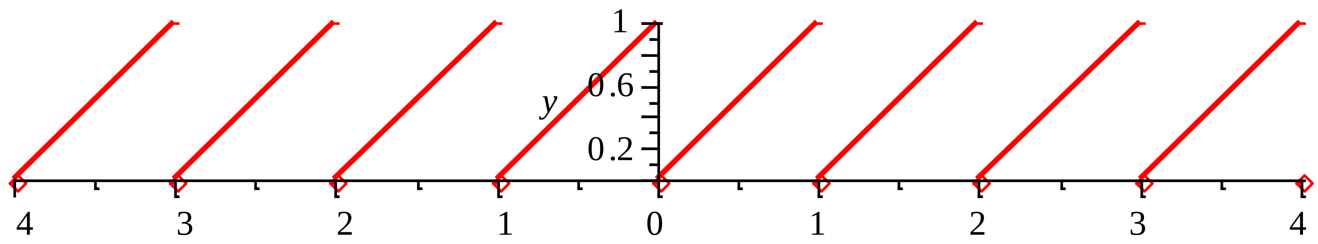
EXERCISE:
Prove ***round(x) = floor(x+0.5) ;***



Errors in Approximation Using floor(x)

If we subtract the ***floor(x)*** from the function ***y = x*** we obtain:

y = x – floor(x).



This equation computes the point by point error caused by approximating a real number ***x*** with an integer number ***floor(x)***. This kind of error is called "***discretization error***" and appears in signal processing and numerical analysis.

If we add up the area under the curve between the x-axis and the red curve we can find the total error that accumulates over an interval [a,b). This idea of finding the area under a curve is called ***integration*** and we will be studying it in detail later.

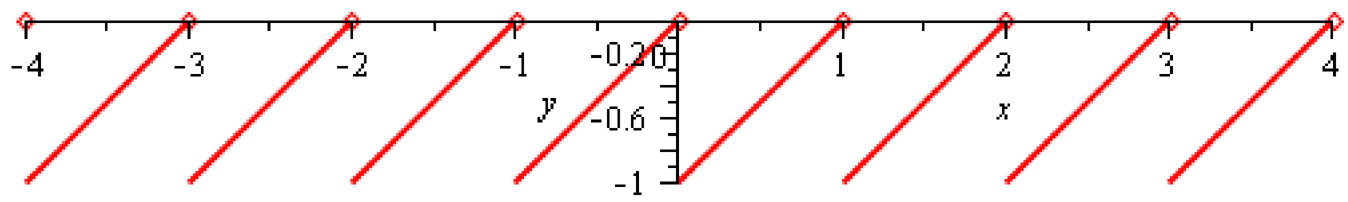
EXERCISE

By adding triangle area, show that the integral of ***x – floor(x)*** for ***x =[0,100)*** is ***50***.

Errors in Approximation Using ceil(x)

If we subtract the function ***y = ceil(x)*** from the function ***y = x*** we obtain:

y = x – ceil(x).



This is the same as the error introduced by approximating ***x*** with ***ceil(x)***. The answer turns out to be the same magnitude (50) as before, but because the area is under the x-axis it is rendered as negative, or -50. So the error is of the same size or *magnitude*, just in a different direction.

EXERCISE

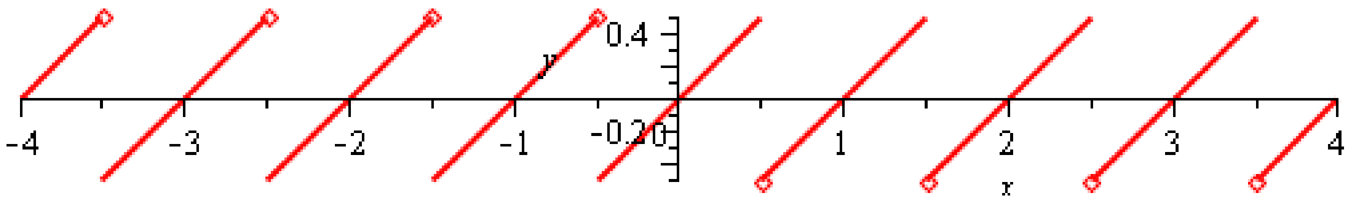
By geometric similarity with previous case,
show the area under the curve of ***x – ceil(x)*** for ***x*** in the interval ***[0,100)*** is ***-50***.

Minimizing Error Using Round(x)

Lecture08-RoundX.gx ×

One case remains. Subtract ***y = round(x)*** from ***y = x*** to obtain:

y = x – round(x).



Because positive and negative errors tend to cancel, we get the remarkable result that the error is 0. This is very important in computing accurate results.

EXERCISE

Show that the area under the curve of ***x – round(x)*** over ***x =[0,100)*** is ***0***.

Which is a better discrete approximation to ***x***, ***floor(x)***, or ***floor(x+0.5)*** and why?

Finding the Limits of Discontinuous Functions

Taking the limit of a discontinuous function is just the same as the continuous case provided one is taking the limit at a value of x in an interval that is continuous. If so all the methods discussed previously apply. In the case where the discontinuous function is closed from the left or right, one may take the left or right-sided limit at that point only. If the function is open on the left or right, the limit is not defined and does not exist.

For example we learned that

Limit of Sum	= Sum of Limits	$\text{limit}(f(x), x \to a) + \text{limit}(g(x), x \to a) = \text{limit}(f(x) + g(x), x \to a)$
Limit of Difference	= Diff. of Limits	$\text{limit}(f(x), x \to a) - \text{limit}(g(x), x \to a) = \text{limit}(f(x) - g(x), x \to a)$
Limit of Product	= Prod. of Limits*	$\text{limit}(f(x), x \to a) \cdot \text{limit}(g(x), x \to a) = \text{limit}(f(x) \cdot g(x), x \to a)$
Limit of Quotient	= Quo. of Limits	$\text{limit}(f(x), x \to a) / \text{limit}(g(x), x \to a) = \text{limit}(f(x) / g(x), x \to a)$

*includes constant case.

To these we add two laws that may also be applied with the care and rigor of those above.

Limit of Power	= Power of Limits	$\text{limit}(f(x)^{n}, x \to a) = \text{limit}(f(x), x \to a)^{n}$
Limit of Root	= Root of Limits	$\text{limit}(f(x)^{1/n}, x \to a) = \text{limit}(f(x), x \to a)^{1/n}$

Limits of Discontinuous Functions

EXERCISE

1) Take the limit of the red function:

y = x, at the value *x = a*.

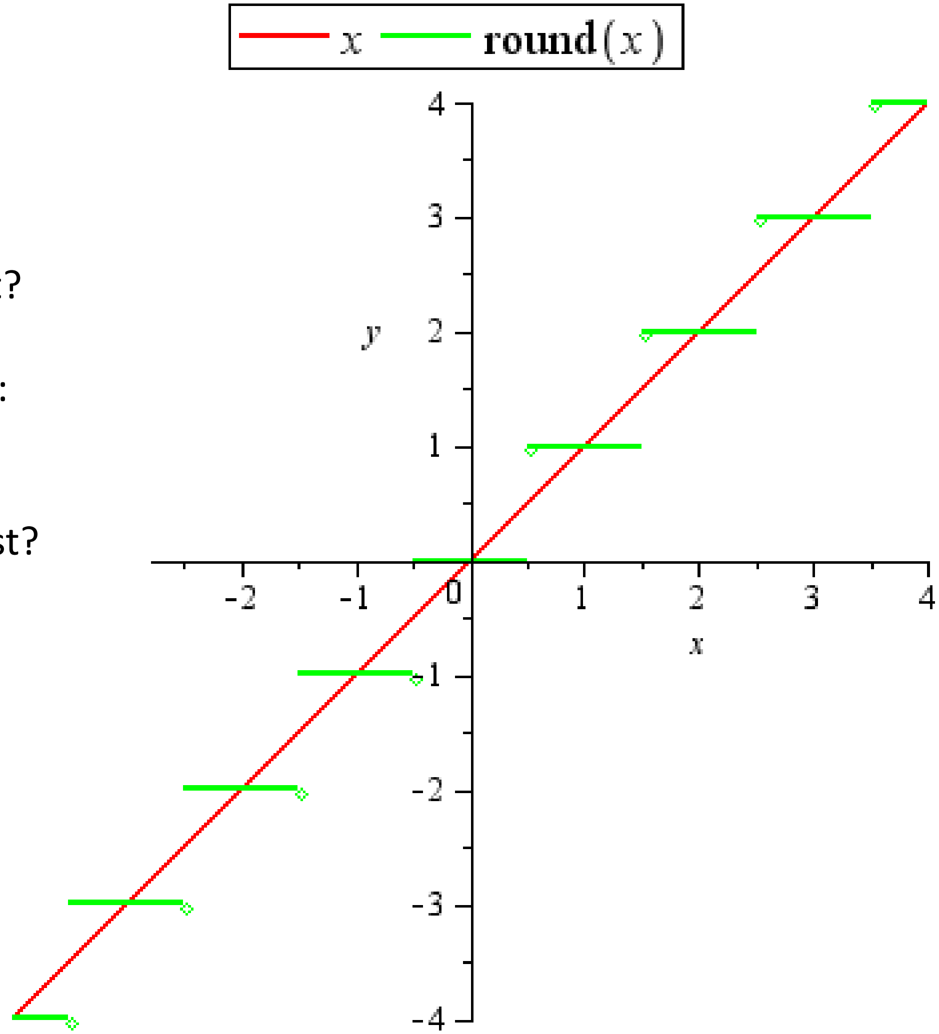
Over what interval does the limit exist?

2) Take the limit of the green function:

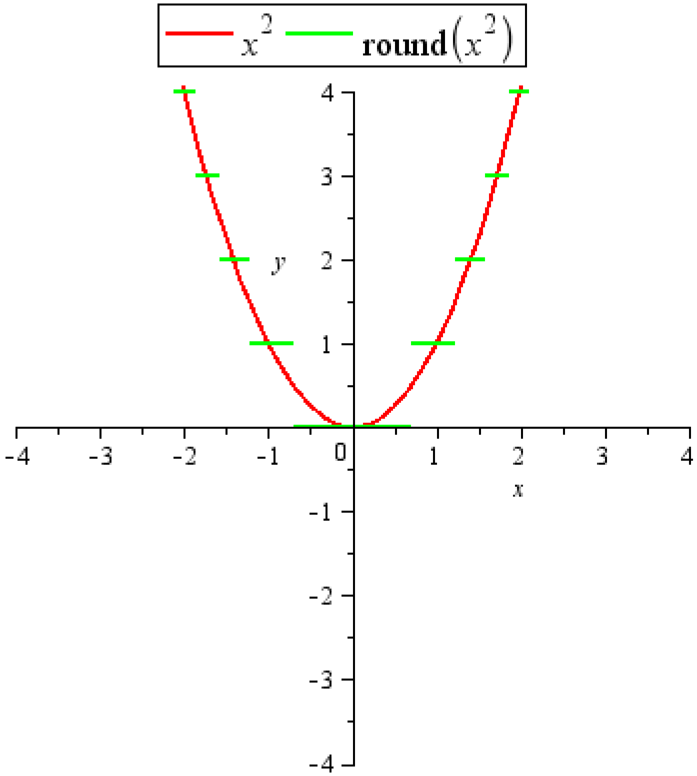
y = round(x)

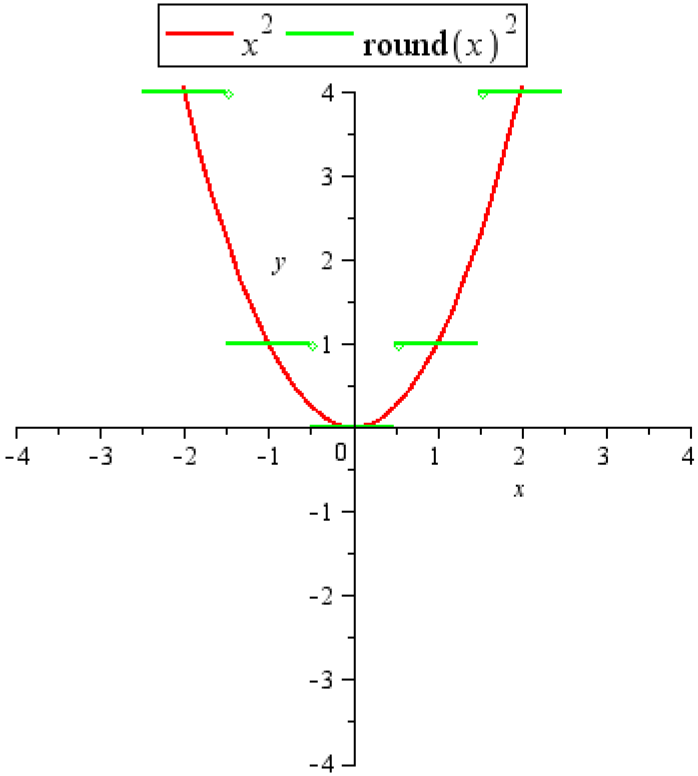
Over what intervals does the limit exist?

At what points is the limit undefined?



Limits of Discontinuous Functions

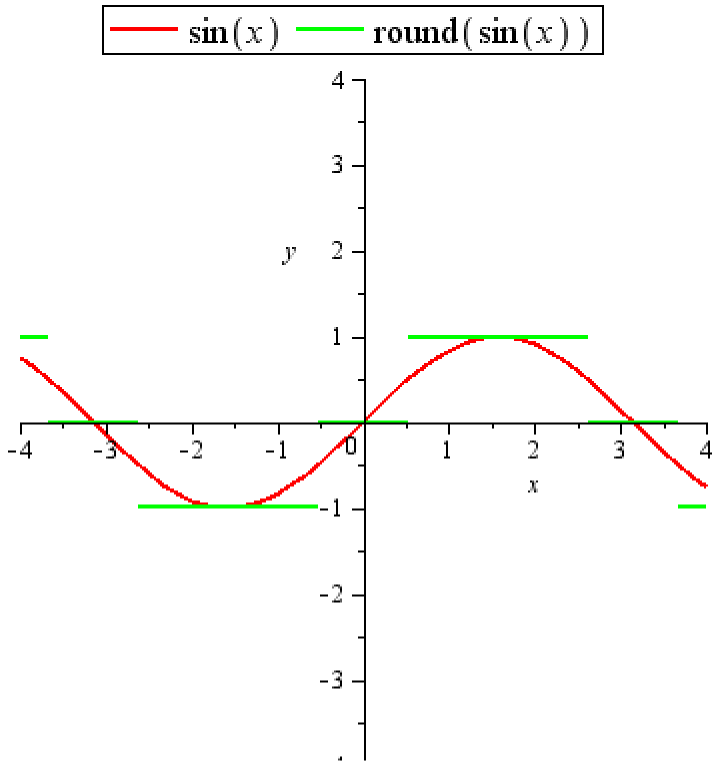


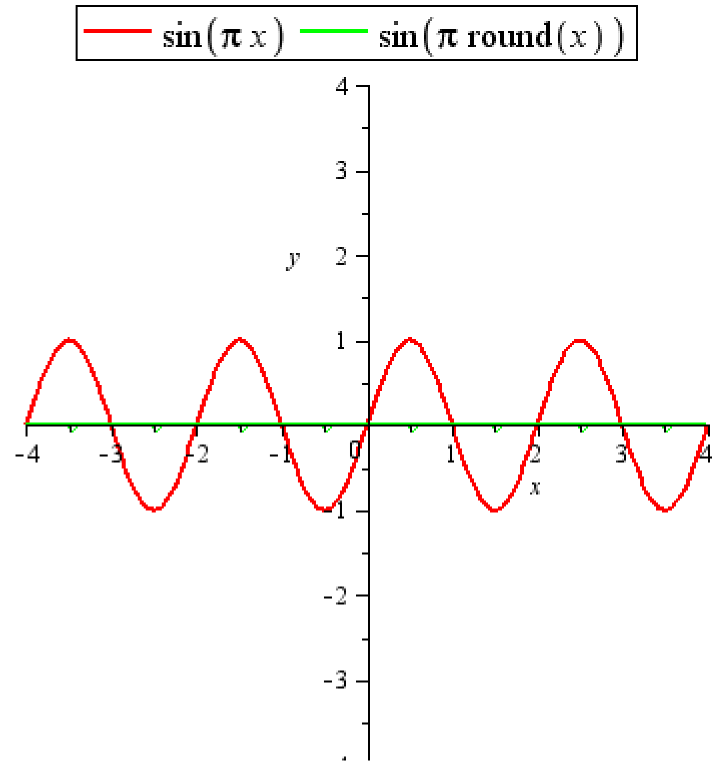


EXERCISE

Take the limit of both pairs of red and green functions.
Over what intervals is the limit defined?

Limits of Discontinuous Functions





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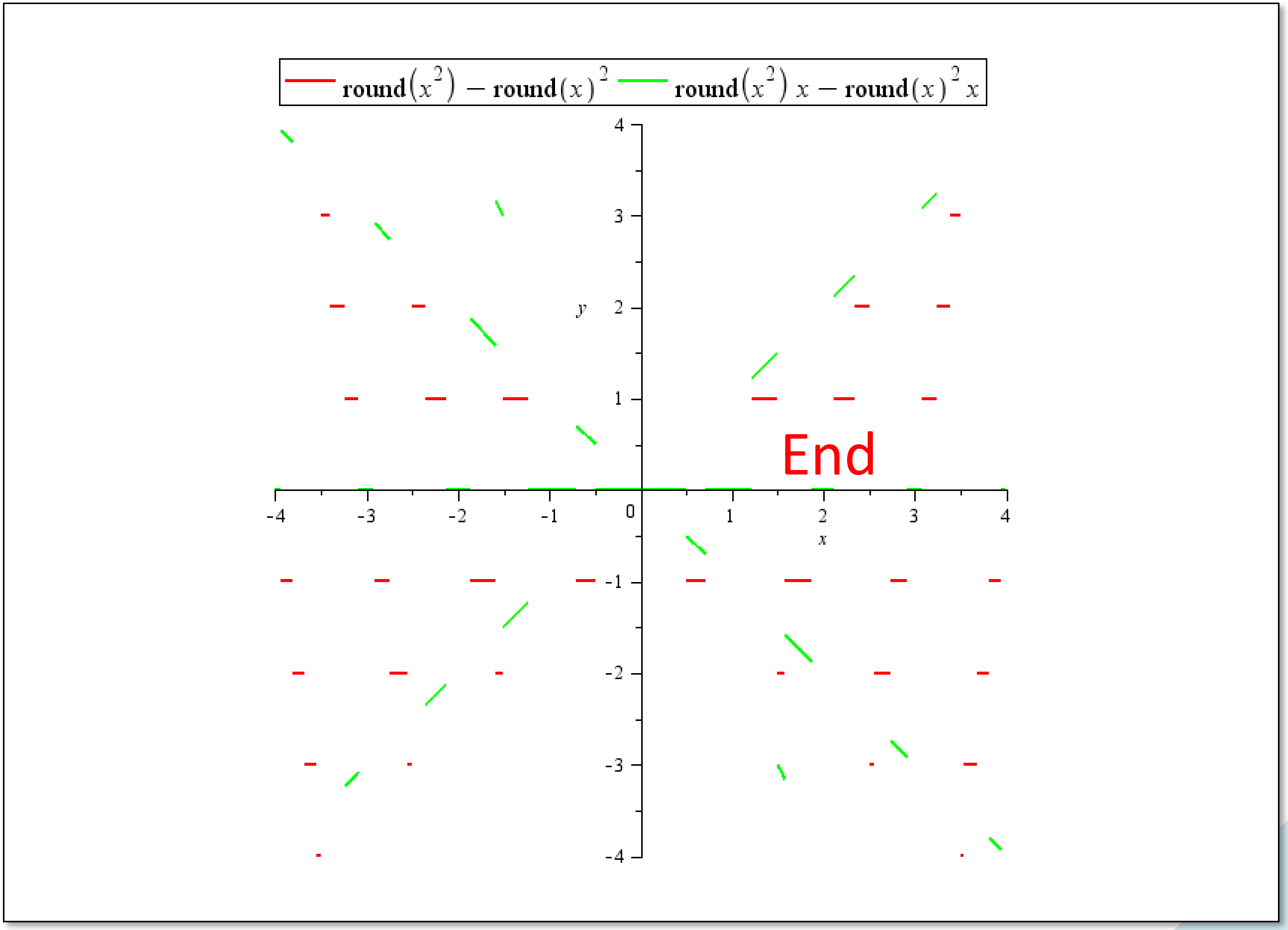
round(x^2) − round(x)^2
round(x^2) x − round(x)^2 x
y
x
End