Learning
Calculus
With
Geometry
Expressions™

by L. Van Warren

L+ε

L

f(a)

L-ε

Drag A
Drag B
Drag C

25

20

15

10

5

-5

-5

5

10

a-δ   a   a+δ

Lecture 9:
Precise Definition of the Limit
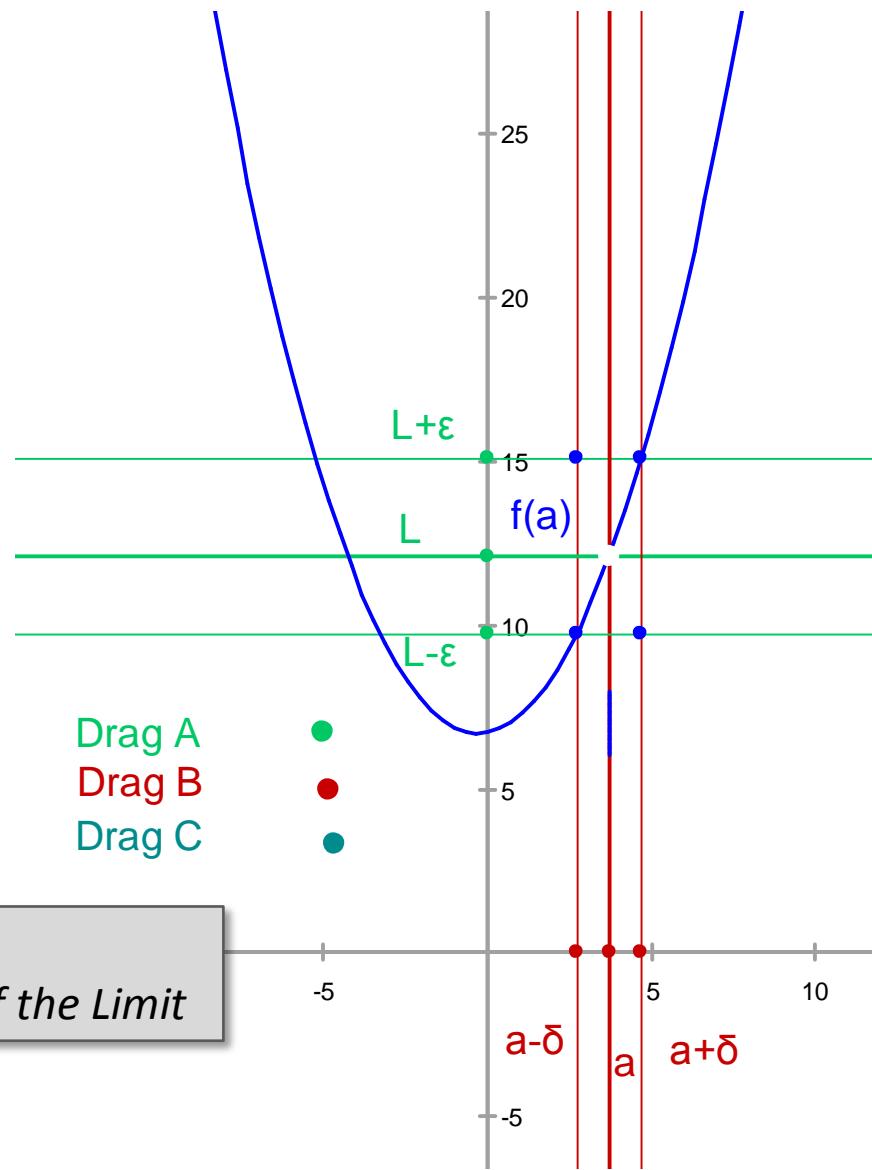
Chapter 2: Limits

Lecture	Topic
5	Finite Limits
6	Infinite Limits
7	Continuity
8	Discontinuity
9	**Precise Definition of The Limit**

Calculus Inspiration

Freek Wiedijk - *"Phrake Weedike"*

PhD. 1991
University of Amsterdam
Thesis: Persistence in Algebraic Specifications

- Automated Theorem Proving

- Seventeen Theorem Provers of the World

- Formalizing 100 Theorems

- Geometry of Supergravity

- Courtesy Nederlands Mathematisch Congress

Calculus Inspiration

Freek Wiedijk is a Dutch computational mathematician who is keeping a list of how many of [Top 100 Mathematical Theorems](#), have been proven by machine.

His contributions are significant because the machine proof of a mathematical theorem eliminates guesswork about hidden assumptions that often characterize human proofs. He has also kept track of which systems have been used to perform the proofs and their relative success.

His master's thesis concerned the geometry of supergravity which he terms "mathematical physics" and his Ph.D. thesis addressed conservativity in modular specifications, or "mathematical computer science" as he puts it.

He has written on comparing automated theorem proving systems and his work is quite relevant to the present topic, "Precise Definition of the Limit", because it is one of the definitions that made calculus possible.

Preamble: Controls and Experiments

In mathematics, science and software:

Positive and Negative Controls
Are Your Best Friend.

- Barry Hurlburt, Ph.D.

Definition: Control

The Expectation of
A Known and Repeatable Output
For a Given Input

Positive Control

Produces a repeatable non-null output for a given non-null input. Something happens, and that something therefore exists, and that something one can observe and measure.

You get the same answer every time you repeat the experiment.

Negative Control

Produces no output or a repeatable null output for a null input.

You still get the same answer every time you repeat the experiment.

Example: Positive and Negative Control

Positive Control:
Enter 2 + 21 on a certain calculator and press =.
Expected result: 23 is displayed.

If you don't get 23, the positive control is informing you that the calculator doesn't work as expected.

Negative Control:
You turn the calculator off.
If it turns on and starts calculating, the negative control is informing you that the calculator doesn't work (and is possibly possessed!)

Experiments Reveal Functional Properties

Mathematicians and scientists devise experiments to discover the functional rules governing systems of interest.

An experiment can be numerical, symbolic, geometric, physical, chemical, mental, etc.

The Precise Definition of the Limit

This true meaning of this definition has traditionally been difficult to teach and to learn. The advent of symbolic geometry makes it much easier to understand. It took 150 years for the definition to be clarified, because several things are happening at once. The definition itself is a functional machine for generating proofs. For a quick and useful introduction to types of proofs see en.wikipedia.org/wiki/Mathematical_proof.

We saw earlier, how the limit process takes a function as input and when substitution is used, produces another function, often simpler, as output.

We will first give the precise definition of the limit in symbolic form. Then we will work through some interactive examples that will clarify exactly what it means:

The limit(f(x), x →a) = L exists iff

Given an ϵ > 0, there exists a δ > 0 such that:

| x − a | < δ implies | f(x) − L | < ϵ

For any x ≠ a in the domain of f.

The Precise Definition of the Limit – cont.

The expression:

limit(f(x), x → a) = L exists iff   given an $\epsilon > 0$, there exists a $\delta > 0$ such that:

| x − a | < δ implies |L - f(x)| < ϵ

For any x ≠ a in the domain of f.

L+ε

L

L-ε

a-δ

a

a+δ
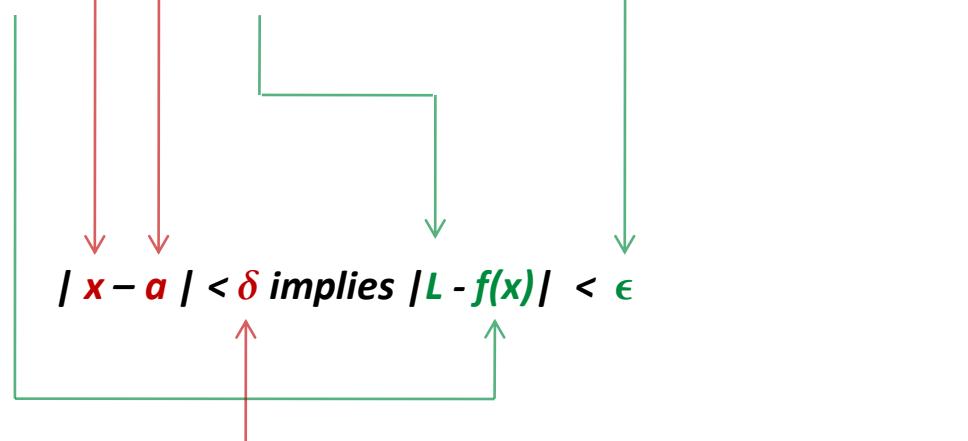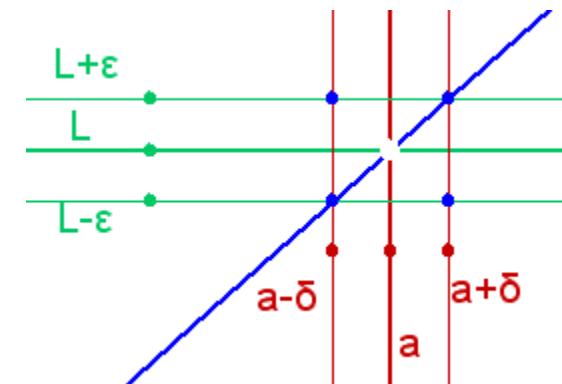
Precise Definition of Limit: PDL

Starting simply, consider the limit of the linear function:

limit(2x+1, x→2) =5

Using substitution we obtain the value 5 for the limit. Proving this is true requires the definition.

EXERCISES:

1) Open the example file shown.

2) Drag the value of **a** and notice how the limit **L** changes.

3) Drag the envelope boundaries **a ± δ** , **L ± ε** and notice how the box bounding the point (**a**, **L**) changes.

f(x)=2x+1

L+ε

L

L-ε

a

a-δ

a+δ

13
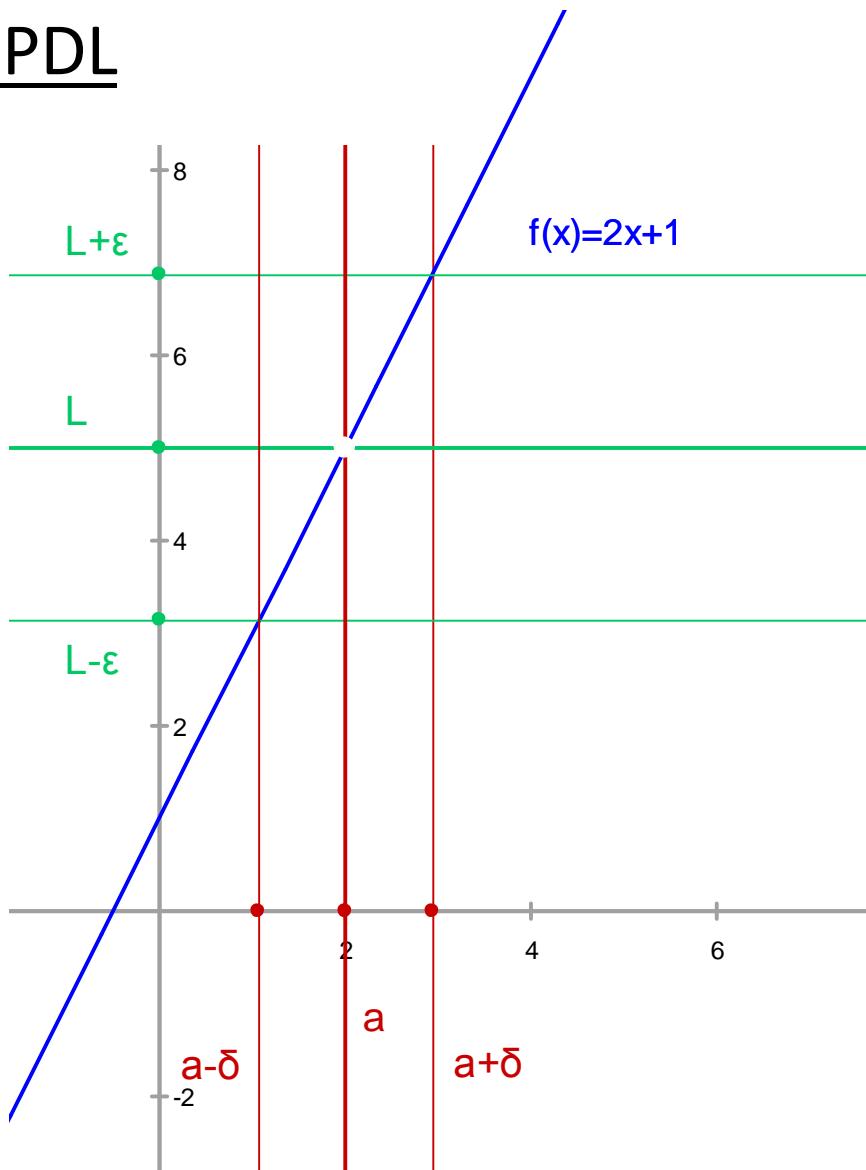
PDL- Spreadsheet

Using a spreadsheet we can build a numerical version of this proof that works for all linear cases. This is shown on the following page. User inputs are highlighted in yellow boxes.

EXERCISES:

1) Open the spreadsheet example file Lecture09-PreciseDefOfLimit1.xls.

2) Change the of slope of f(x) = **2**x+1 from **2** to **3**.
Notice how the change percolates through the proof. Note especially how δ and ϵ are related. This relationship is crucial for the meaning of the proof.

3) Change the b-value or y-intercept of f(x) = 3x+**1** from **1** to **0**.
Does the constant's value change the ϵ, δ relationship. Why is this so?

4) Change the limit value from x→2 to x→4.
Does this affect the values in the proof and the relationship between δ and ϵ ?

5) Change the variable x to the variable t. What happens to the proof?

Definition

The limit (f (x) , x → a) = L   exists iff

Given $\epsilon > 0$ there exists $\delta > 0$

such that | x - a | < δ $\Longrightarrow$ | f   (x   ) - L | < ϵ

Prove

The limit (2 x + 1 , x → 2) = 5   exists

Let $\epsilon > 0$ Find $\delta > 0$

such that | x - 2 | < δ $\Longrightarrow$ | f   (x   ) - 5 | < ϵ

| (2   x + 1) - 5 | < ϵ

| 2   x + 1 - 5 | < ϵ

| 2   x   - 4 | < ϵ

| 2   (x   - 2) | < ϵ

| 2 | | (x   - 2) | < ϵ

2 | x   - 2 | < ϵ

Let   2 δ = ϵ

∴ δ = ϵ / 2

if | x - 2 | < δ

then | x - 2 | < ϵ / 2 $\Longrightarrow$ | f   (x   ) - 5 | < ϵ

QED ∎

15

Precise Definition of Limit

We can generalize the linear function to:

limit(mx+b, x→a) =L

EXERCISES:

1) Open the example file shown.

2) Drag the value of **a** and notice how the limit **L** changes.

3) Drag the envelope boundaries **a ± δ** , **L ± ε** and notice how the box bounding about the point (**a**, **L**) changes.

4) Change the values of **m** and **b** by dragging them.

L+ε

L

f(a)

L-ε

b

f(x)=mx+b

m

a-δ

a

a+δ
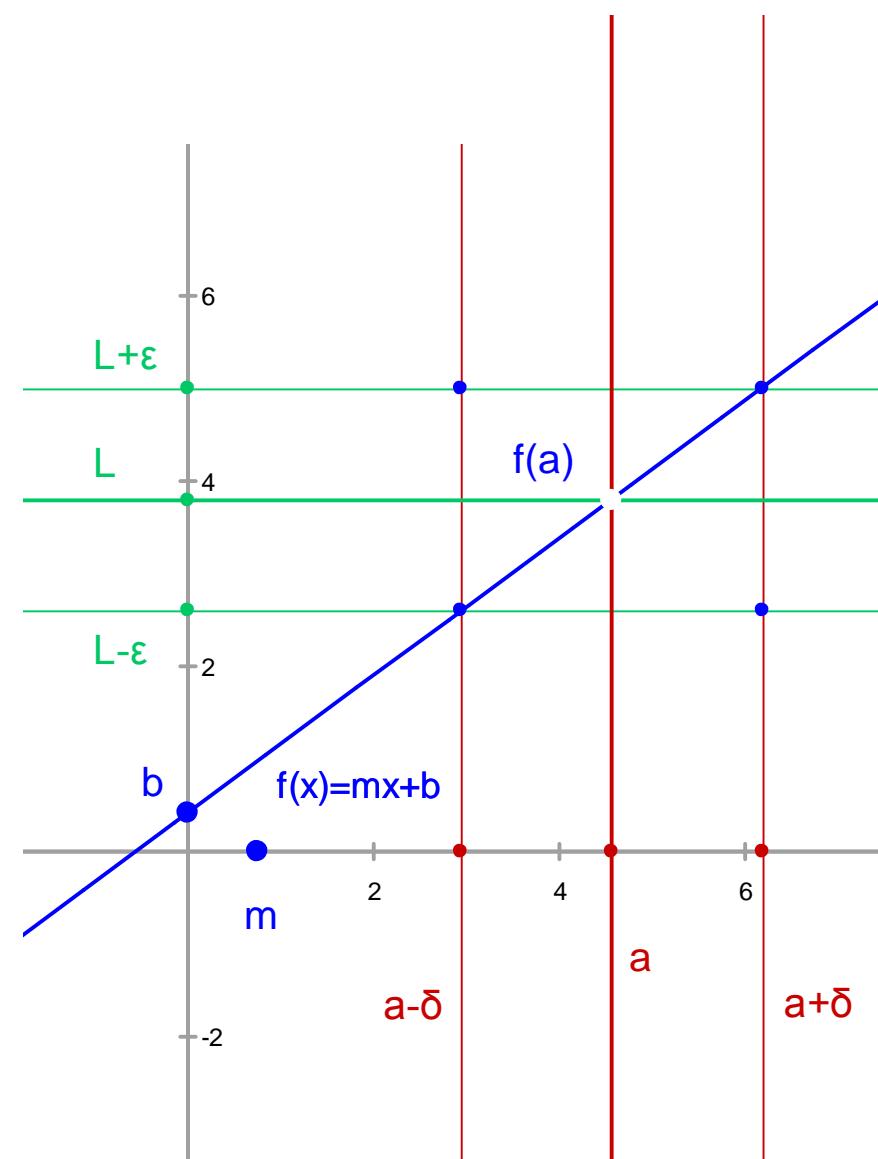
PDL as Bounding Box Recipe

In reality, the PDL provides a **bounding box** around the point (**a**, **L**).
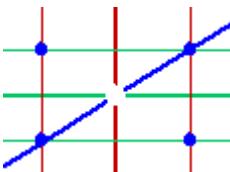
The delta-epsilon machine provides two essentials:

1) A method for finding this bounding box.
2) The assertion that if the bounding box can be found for all δ and ϵ that the limit exits.

The "trick" to PDL is solving the expression | $f(x) - L$ | for $|x - a|$.

When we do, it is possible to express ϵ in terms of δ, and vica versa. The resulting solution enables us to draw the sides of the bounding box. *If you can draw it, you can understand it.*

We have worked through the linear case, now let's spreadsheet a quadratic form:

The limit(A + B x + C x², x → a) = L

We will modify our Geometry Expressions machinery and then do some examples.

PDL: General Quadratic Case

We can generalize the linear function to a quadratic by adding a third term. The coefficient names are written in upper case. We write:

$$Y = A + B\,x + C\,x^2$$

Looking at the graph, notice
The x-intervals of $a \pm \delta$ are
equally spaced, while
the intervals in y of $L \pm \epsilon$ are not.

EXERCISE

Drag the values of A, B and C and notice how the shape of the curve changes. Then drag the δ and ϵ bounds and note how the bounding box changes. (Drag slowly!)

Drag A ●
Drag B ●
Drag C ●

L+ε
L
f(a)
L-ε

a-δ   a   a+δ

-10   -5   5   10

25
20
15
10
5
-5
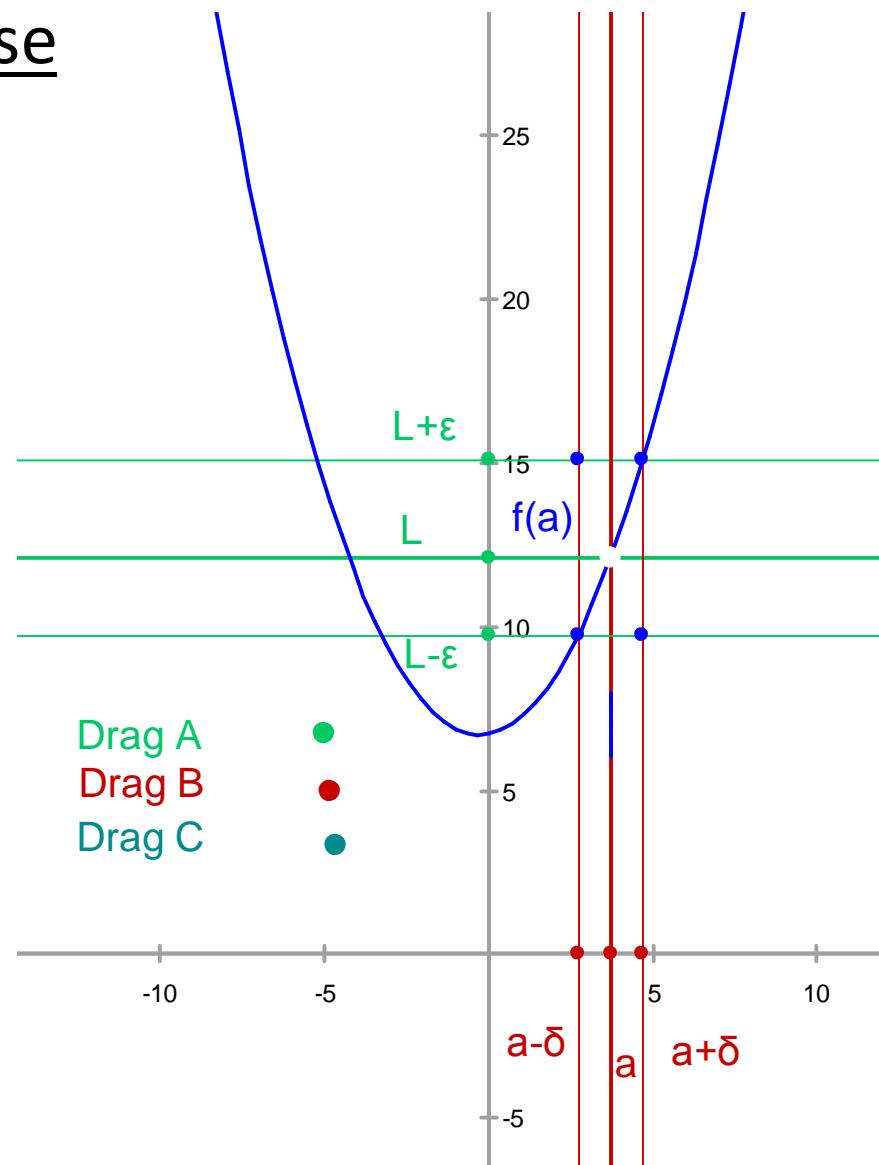
PDL- Quadratic Spreadsheet

We now construct a numerical version of the quadratic proof that works for all real cases. This is shown on the following page.

EXERCISES:

1) Open the spreadsheet example file Lecture09-PreciseDefOfLimit2.xls.

2) Change the of values of the coefficients, A, B and C.

3) Some values of A, B and C result in spreadsheet errors, plot these cases in Geometry Expressions™ and state why the errors occur. How might they be eliminated?

Definition The limit (f ($\boxed{x}$) , x → a) = L

Exists iff given ϵ > 0   there exists δ > 0

such that | x - a | < δ

$\Longrightarrow$ | f (x) - L | < ϵ

Prove The limit ($\boxed{2}$ x ^{2} + $\boxed{4}$ x + $\boxed{-2}$, x → $\boxed{2}$) = 14 exists.

Let ϵ > 0   Find δ > 0

such that | (x - 2) | < δ

$\Longrightarrow$ | 2 x ^{2} + 4 x + -2 - 14 | < ϵ

scaling | x ^{2} + 2 x + -1 - 7 | < ϵ / 2

simplifying | x ^{2} + 2 x   - 8 | < ϵ / 2

factoring | (x - -4)   (x - 2) | < ϵ / 2

| (x - -4) | | (x - 2) | < ϵ / 2

substituting | (x - -4) | δ < ϵ / 2

we showed previously   x → 2 ,

implies | (x - -4) | → 6 δ < ϵ / 2

∴ δ < ϵ / 12

if | x - 2 | < δ

then | x - 2 | < ϵ / 12 $\Longrightarrow$ | f (x) - 14 | < ϵ ∎

PDL: Exploring Linear and Quadratic Cases

Symbolic geometry spares us the suffering expended over the years grappling with PDL. To complete your understanding use the examples to investigate the following.

EXERCISES

1) limit(5, x → 7) = 5

2) limit(b, x → a) = b

3) limit(a, x → b) = a

4) limit(x, x → 3) = 3

5) limit(x, x → c) = c

6) limit(a · x, x → 3) = a · 3

7) limit(a · x, x → b) = a · b

8) limit(m · x, x → c) = m · c

9) limit((x + 3), x → a) = a + 3

10) limit((x + b), x → a)) = a + b

11) limit((x + a), x → b)) = b + a

12) limit((x + a), x → b) = limit((x + b), x→a)

13) limit(2 (x + a), x → b) = 2(b + a)

14) limit (k (x + a), x → b) = k(b + a)

15) limit ((x + 1) (x + 2), x → a) = (a + 1) (a + 2)

16) limit ((x + a) (x + b), x → c) = (c + a) (c + b)

17) limit ((x + 1) (x - 2), x → a) = (a + 1) (a - 2)

18) limit ((x + a) (x - b), x → c) = (c + a) (c - b)

The Triangle Inequality: Scalar

The triangle inequality states that the length of any side of a triangle is less than the length of the two remaining sides. This can be written:

$$c < a + b$$

It is a restatement of the idea that the shortest distance between any two points is a straight line.

Notice that the length of a side is never negative.

EXERCISES:

1) Open the example file and drag a gray dot.

2) Its it possible to make c longer than a + b?

3) When c = a + b do you still have a triangle?
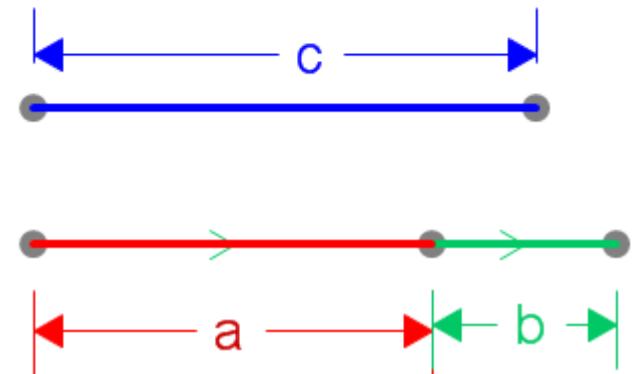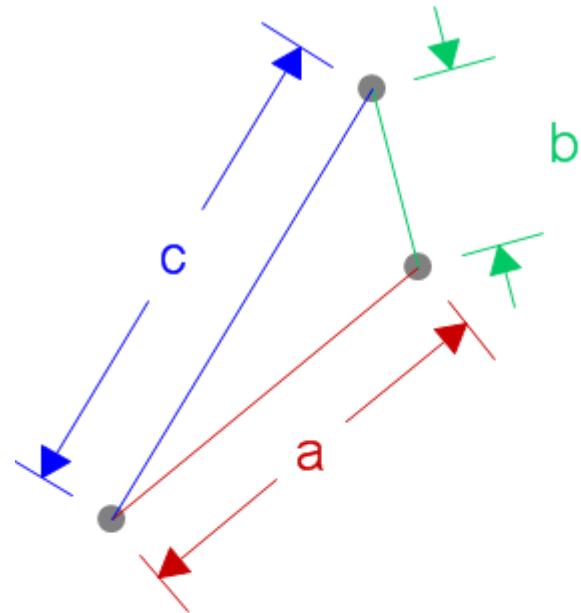
The Triangle Inequality: Vector

Vectors each with an (x, y) component can be added by placing them head to tail.

Here the vector A is added to B to give C such that

$$C = A + B$$

The length of a vector can be written as :

$$\text{length}(C) \quad OR \quad \| C \|$$

We can restate the triangle inequality as:

$$\| C \| \leq \| A \| + \| B \|$$

But we can substitute $C = A + B$ to obtain:

$$\| A + B \| \leq \| A \| + \| B \|$$

C

B

A

||A||

||B|| ||C||

23
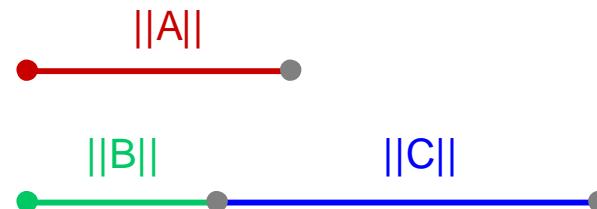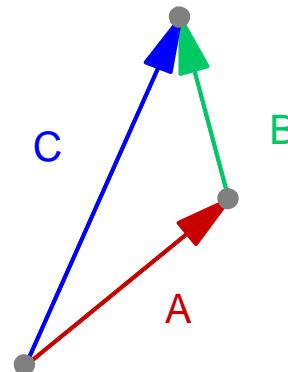
The Absolute Value: An Identity

For any two real numbers , a and b :

$$|a + b| \leq |a| + |b|$$

Note the similarity of this expression to the previous one. There is a difference however, a and b can be negative. For the result to be meaningful a or b must be negative, otherwise the identity degenerates to:

$$a + b = a + b$$

The identity is often called the Triangle Inequality, even though it has nothing to do with a triangle, as the lengths of a triangle's sides are always positive.

EXERCISES:

1) Open the example file shown and drag any gray dot.
2) Is it possible to make $|a + b|$ longer than $|a| + |b|$?
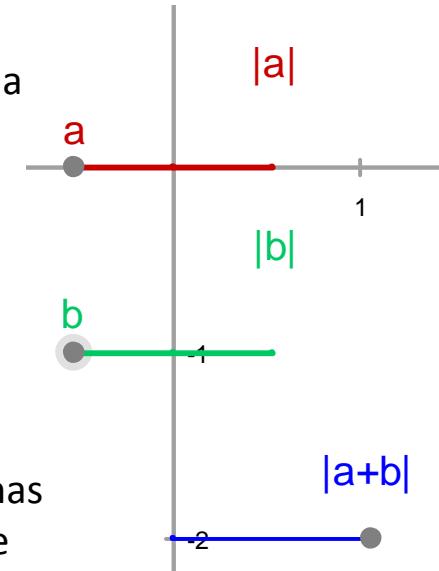
Using PDL for Piecewise Functions

Problem:

f(x) = x   if x < 2
        x + b   if x ≥ 2

What is the **limit(*f(x)*, *x→2*) *as b varies*?**

EXERCISES:

1) Open the example file shown.

2) Drag the envelope boundaries
   a ± δ , **L ± ε** and notice how the box
   bounding about the point (**a**, **L**) changes.

3) Use the PDL and the triangle inequality to
   show those cases for which the limit exists.

L+ε

L

L-ε

a-δ

a

a+δ

6

4

2

2

4

6

-2

25
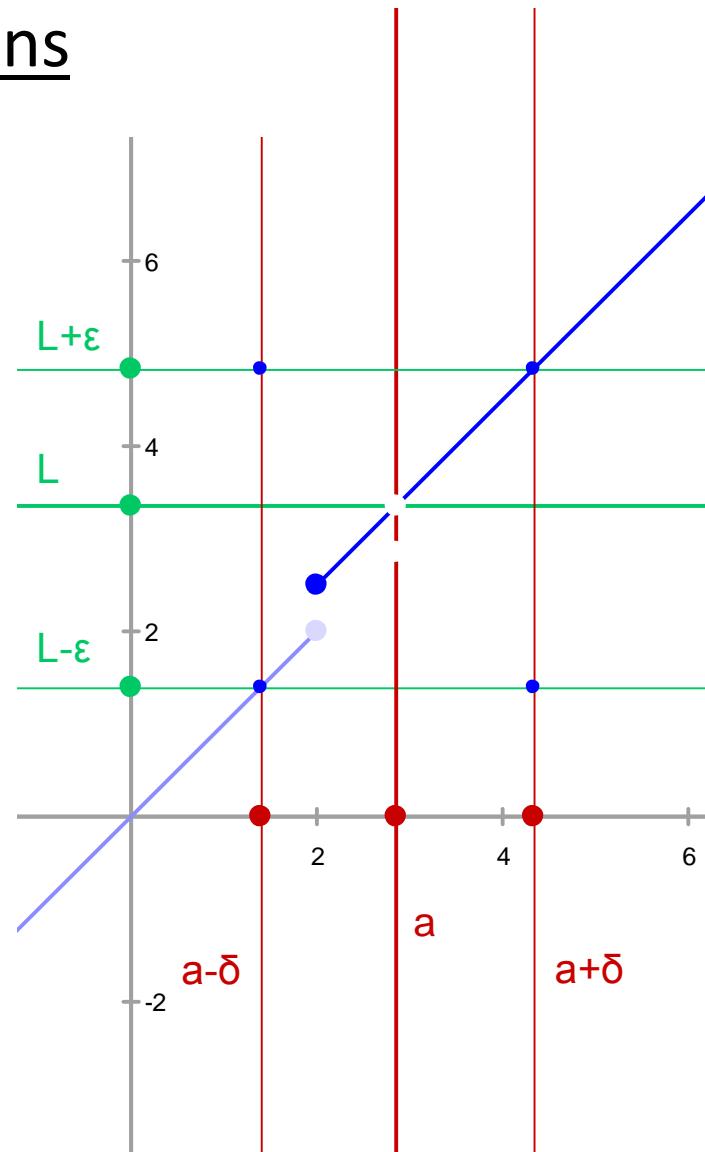
Sketch of A Proof By Contradiction

1.   Assume L exists.
2.   Then for each $\epsilon > 0$ $\exists$ $\delta > 0$:
3.   If $0 < |x-1| < \delta$ then $|f(x)- L| < \epsilon$.
4.   Find one $\epsilon > 0$ such that no $\delta > 0$ works.
5.   Contradiction, Statement 1 is false!
6.   Therefore L cannot exist.

$\exists$ - Stands for "there exists"

End