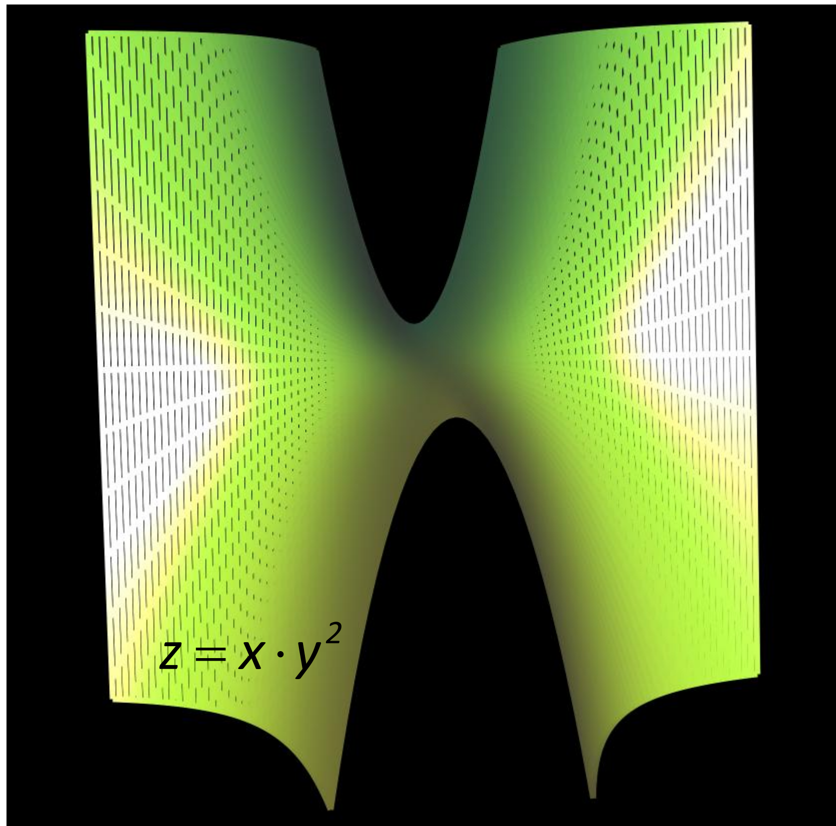
Learning Calculus With Geometry Expressions™

by L. Van Warren

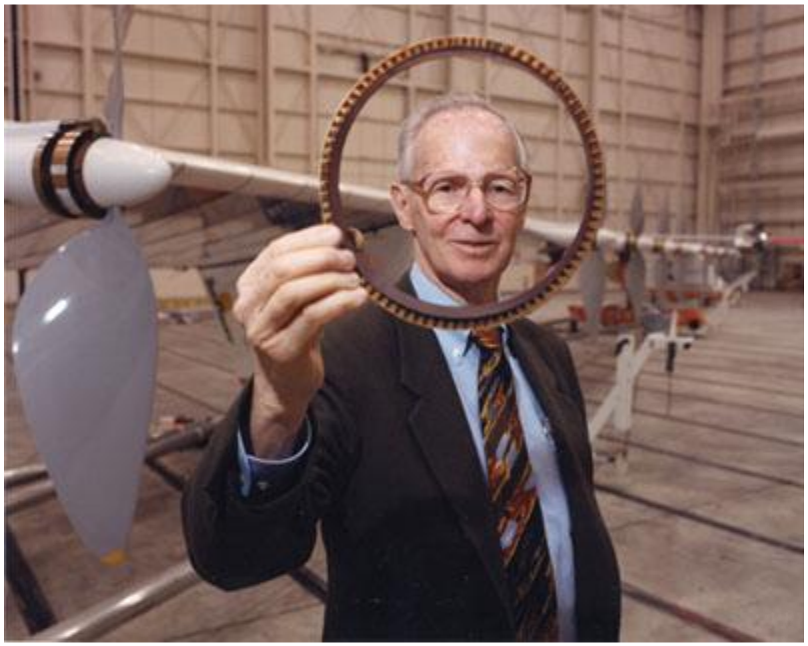
Lecture 16: Extrema, Max & Min



Chapter 4: Rates and Extremes

LECTURE	TOPIC
14	*RATES OF CHANGE*
15	*RELATED RATES*
16	***EXTREMA – MAXIMA AND MINIMA***
17	*DERIVATIVE TESTS AND MEAN VALUE THEOREM*
18	*OPTIMIZATION*

Inspiration



- Photo courtesy NASA

Dr. Paul MacCready *(1925 – 2007)*

At 15, Won a National Aviation Contest

Ph.D. CalTech
Founded AeroVironment™

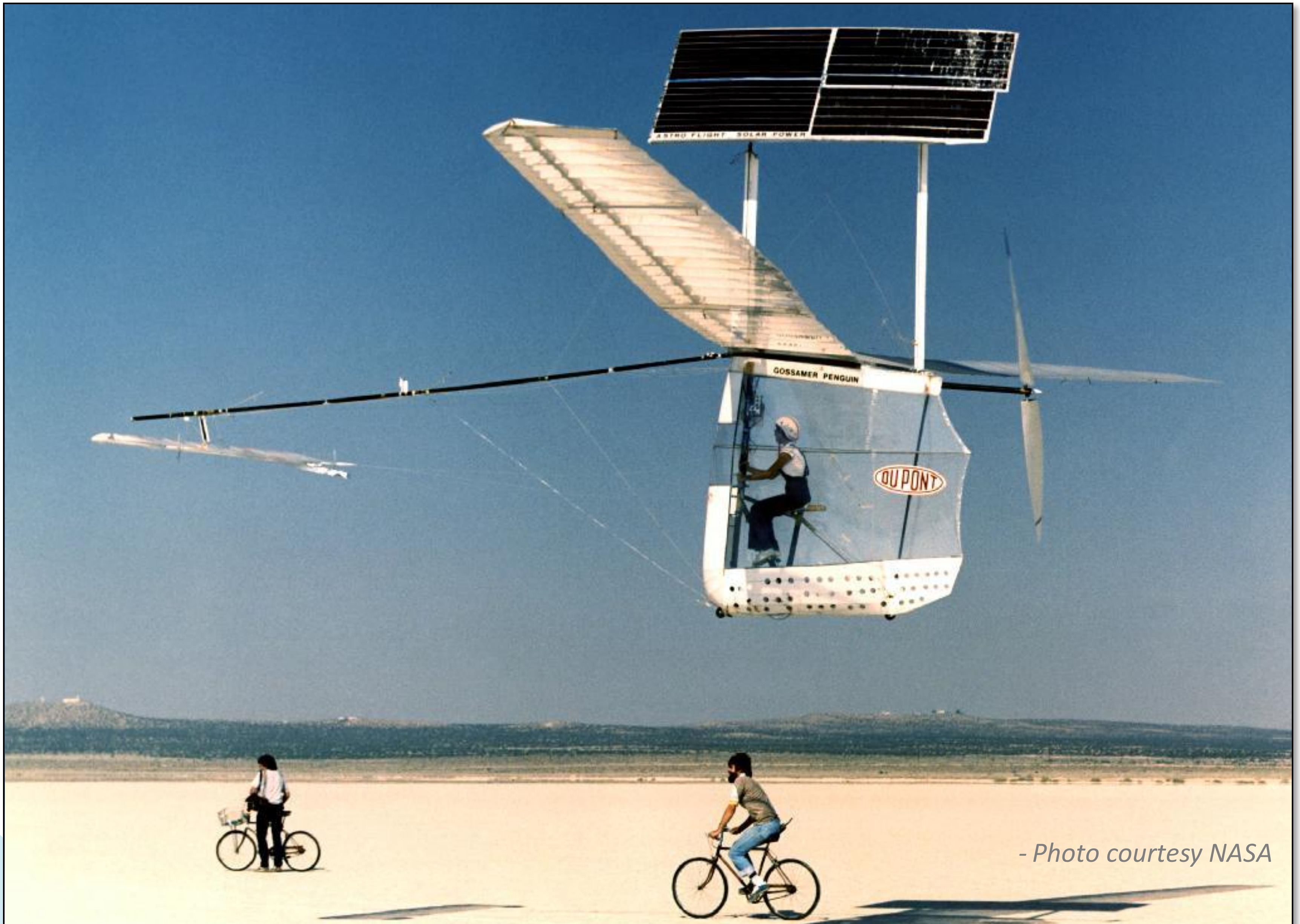
First Human Powered Airplanes
- Gossamer Condor
- Gossamer Albatross*

First Practical Electric Cars
- SunRaycer
- Impact

First Solar Aircraft
- Gossamer Penguin*

* pictured below


Gossamer Albatross
DUPONT
- © Don Monroe – with permission



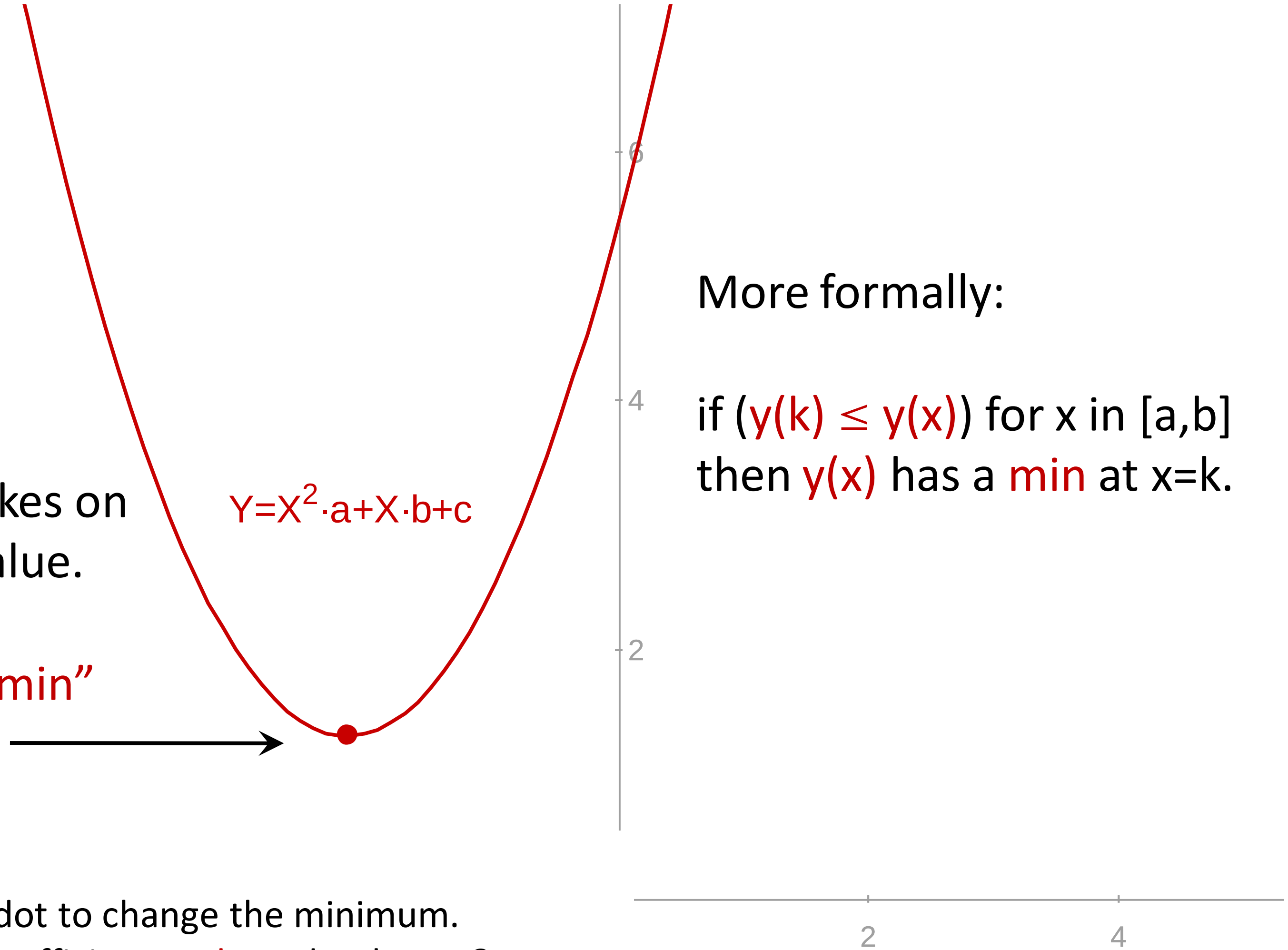
- Photo courtesy NASA

Lecture16-FindingMaxMin.gx

Minimum of a Function

At the red dot
the function takes on
its minimum value.

We call this a "min"
for short.



More formally:

if ($y(k) \leq y(x)$) for x in [a,b]
then $y(x)$ has a min at x=k.

Exercise:

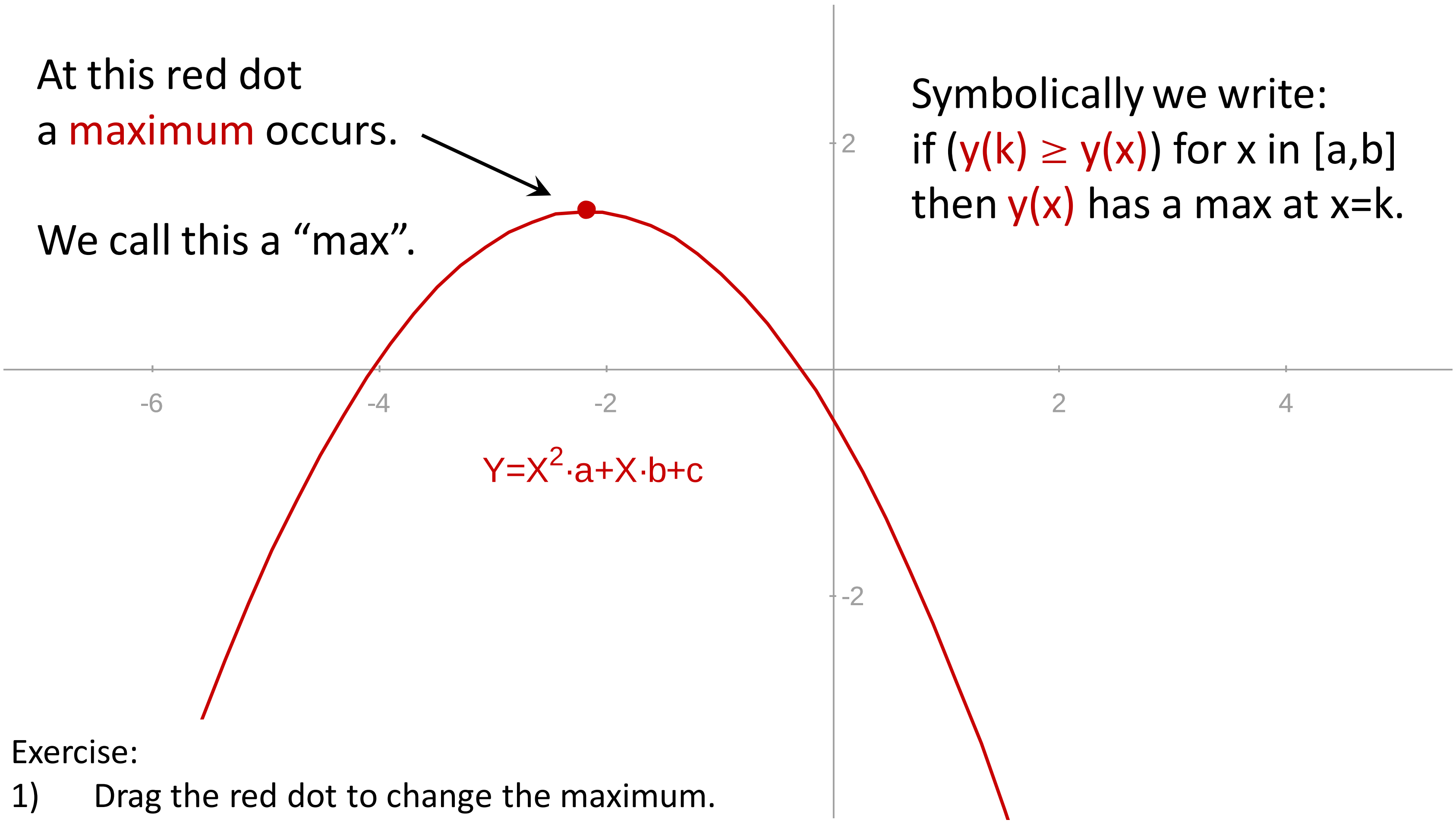
1) Drag the red dot to change the minimum.
2) How do the coefficients a, b, and c change?

Lecture16-FindingMaxMin.gx

Maximum of a Function

At this red dot
a maximum occurs.

We call this a "max".



Symbolically we write:
if ($y(k) \geq y(x)$) for x in [a,b]
then $y(x)$ has a max at x=k.

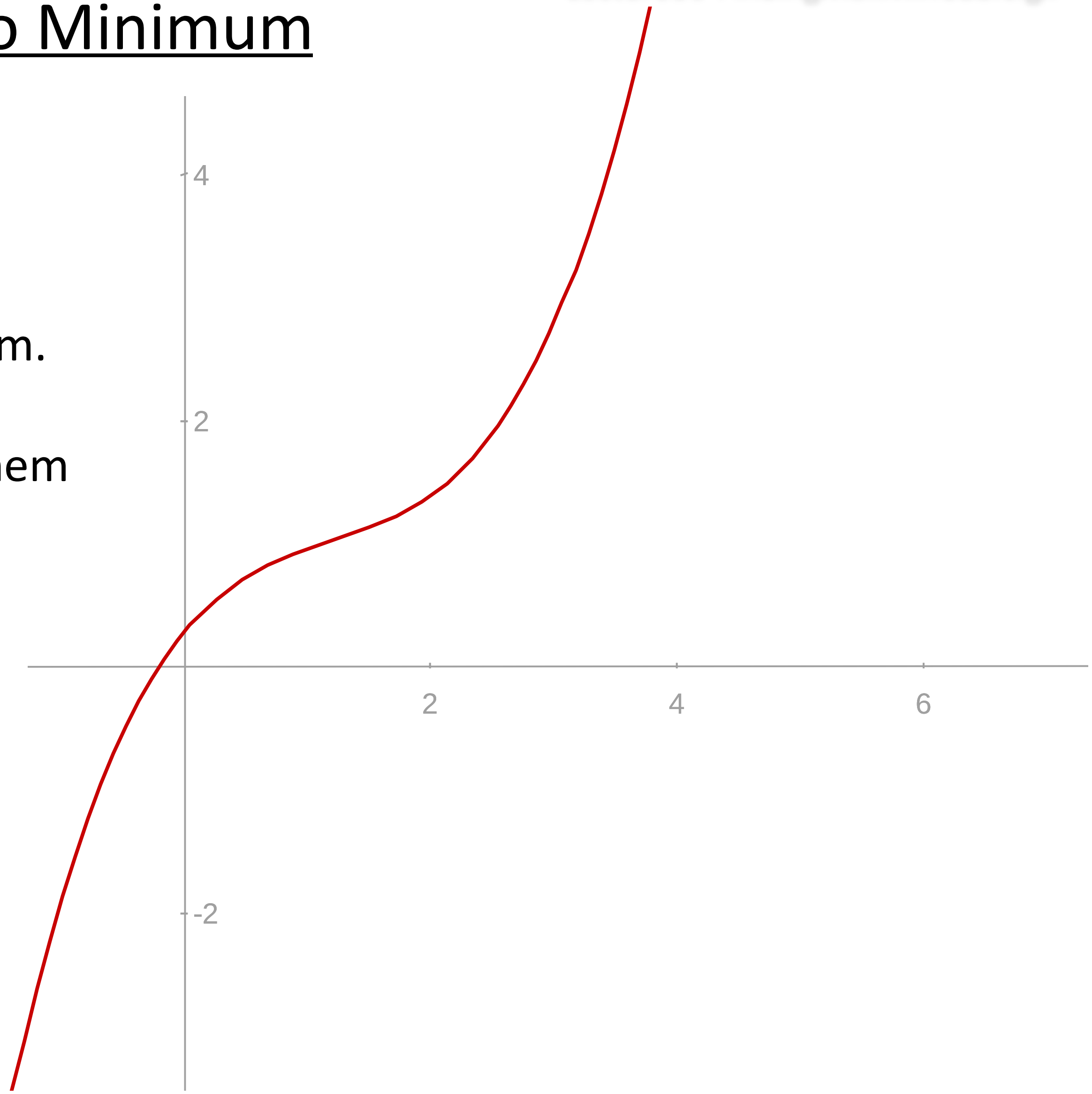
Exercise:
1) Drag the red dot to change the maximum.
2) Use View→Toggle Hidden to reproduce the view shown here.

Lecture16-FindingMaxMinCubic.gx

No Maximum, No Minimum

Considering ∞, some functions have no maximum or minimum.

We can always box them in however.



Exercise:

1) Open the example.
2) Select View→Show All.
3) Show that the first derivative is never zero anywhere on this curve.

Lecture16-FindingMaxMin.gx

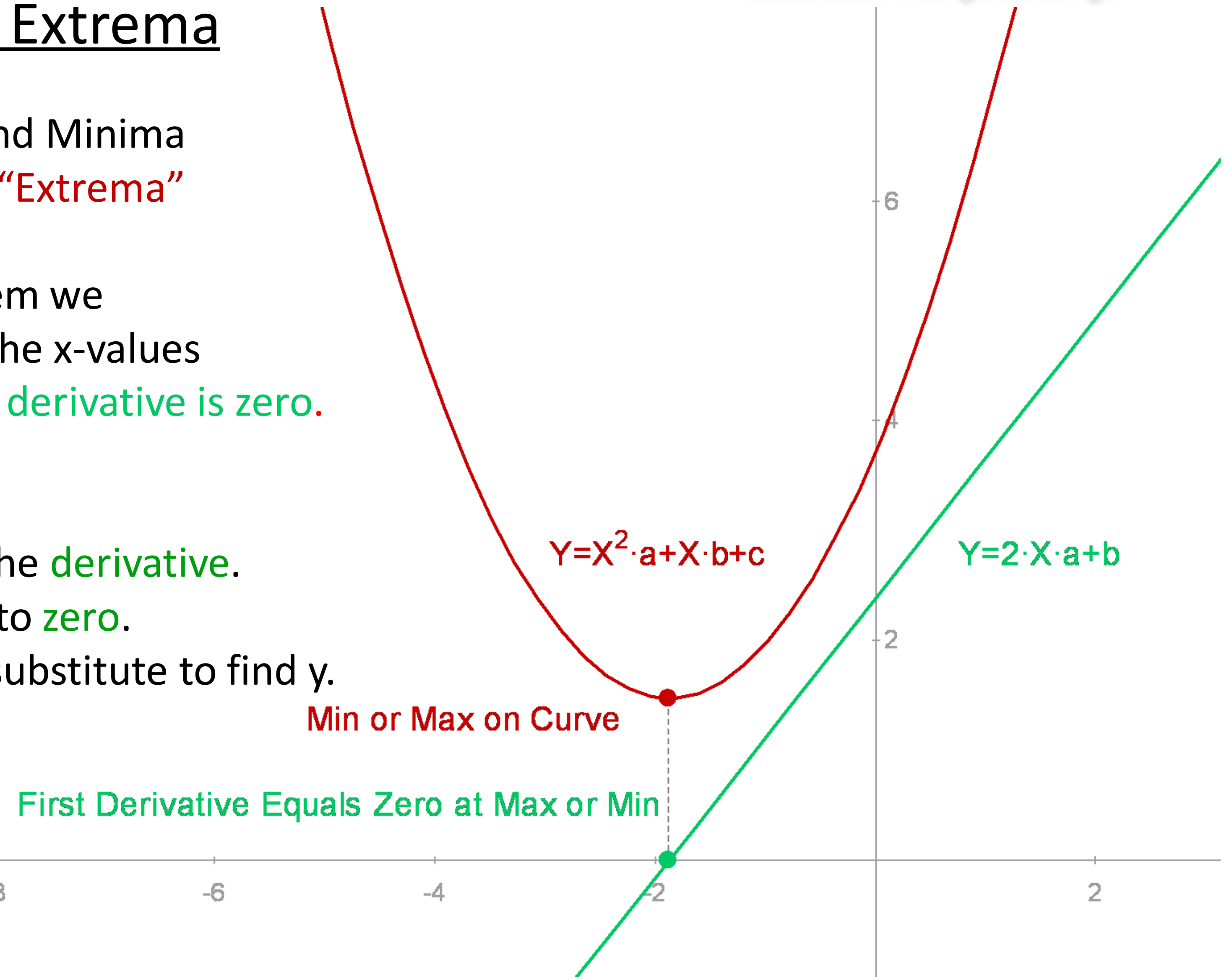
Finding Extrema

Maxima and Minima
are called "Extrema"

To find them we
compute the x-values
where the derivative is zero.

Method:

1) Find the derivative.
2) Set it to zero.
3) Back substitute to find y.



Extremes of a Function: wxMaxima™

```
y(x):=a*x^2+b*x+c;
```

$$y(x):=a\,x^2+b\,x+c$$

1) Write the equation.

```
diff(y(x),x);
```

$$2\,a\,x+b$$

2) Differentiate.

```
solve(%,x);
```

$$[\,x=-\frac{b}{2\,a}\,]$$

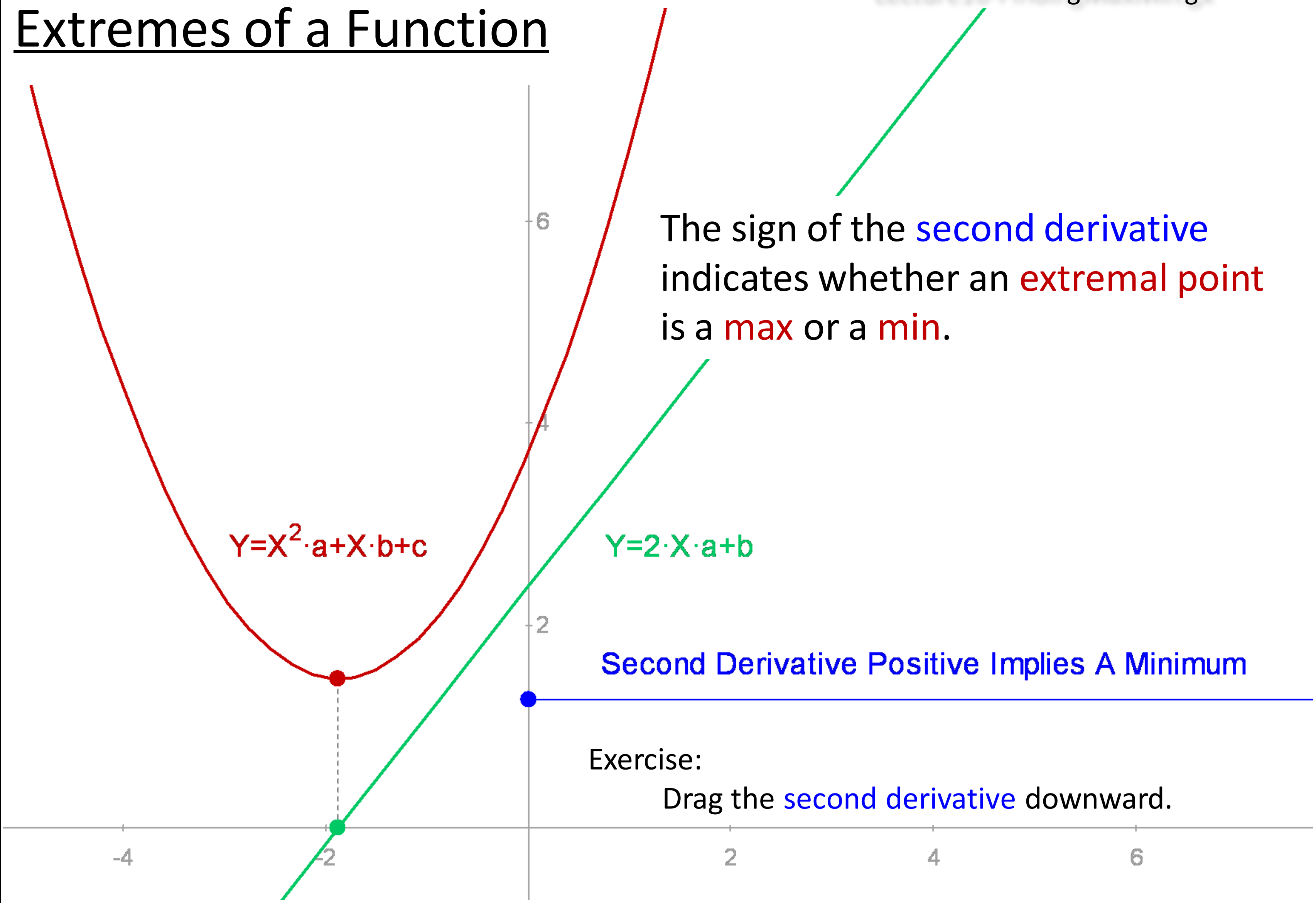
3) Set derivative to zero. Solve for x.

```
y(-b/(2*a));
```

$$c-\frac{b^2}{4\,a}$$

4) Back substitute to find minimum y.

Extremes of a Function



The sign of the second derivative indicates whether an extremal point is a max or a min.

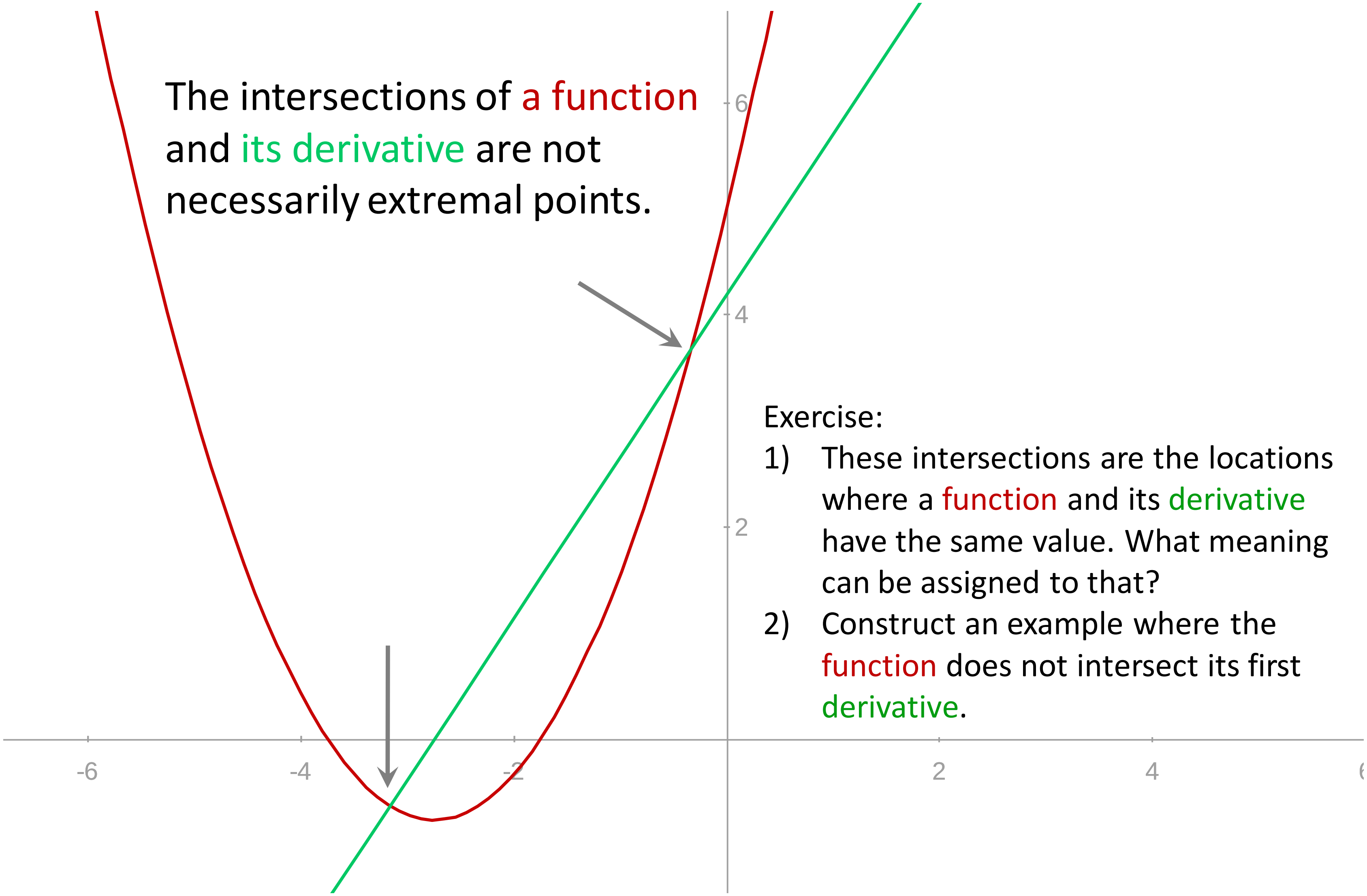
Exercise:

Drag the second derivative downward.

Lecture16-FindingMaxMin.gx

Intersection of Function and Derivative

The intersections of a function and its derivative are not necessarily extremal points.



Exercise:

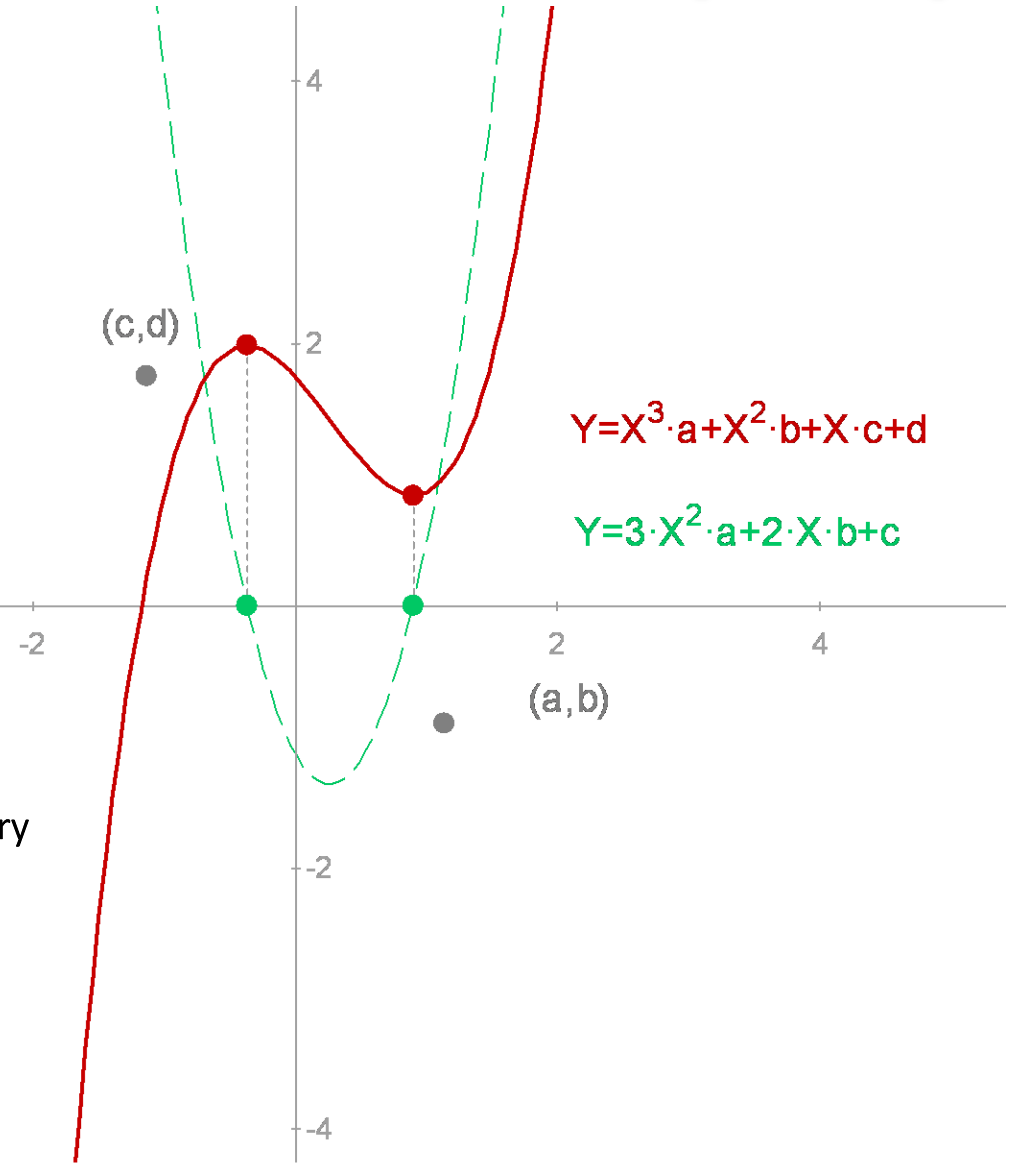
1) These intersections are the locations where a function and its derivative have the same value. What meaning can be assigned to that?
2) Construct an example where the function does not intersect its first derivative.

Lecture16-FindingMaxMinCubic.gx

Local Extrema

Some functions have a local maximum or minimum.

Some have both!



Exercise:

1) Open the example.
2) Drag the gray control points to vary coefficients, a, b, c and d.
3) Record what happens.

Extremes of a Function: Cubics

Lecture16-FindingMaxMinCubic.wxm

```
y(x):=a*x^3+b*x^2+c*x+d;
diff(y(x),x);
solutions : solve(%,x);
```

$$y(x):=a\,x^3+b\,x^2+c\,x+d$$

$$3\,a\,x^2+2\,b\,x+c$$

$$[\,x=-\frac{\sqrt{b^2-3\,a\,c}+b}{3\,a},\ x=\frac{\sqrt{b^2-3\,a\,c}-b}{3\,a}\,]$$

```
y(second(first(solutions)));
```

$$d-\frac{\left(\sqrt{b^2-3\,a\,c}+b\right)^3}{27\,a^2}+\frac{b\left(\sqrt{b^2-3\,a\,c}+b\right)^2}{9\,a^2}-\frac{c\left(\sqrt{b^2-3\,a\,c}+b\right)}{3\,a}$$

```
y(second(second(solutions)));
```

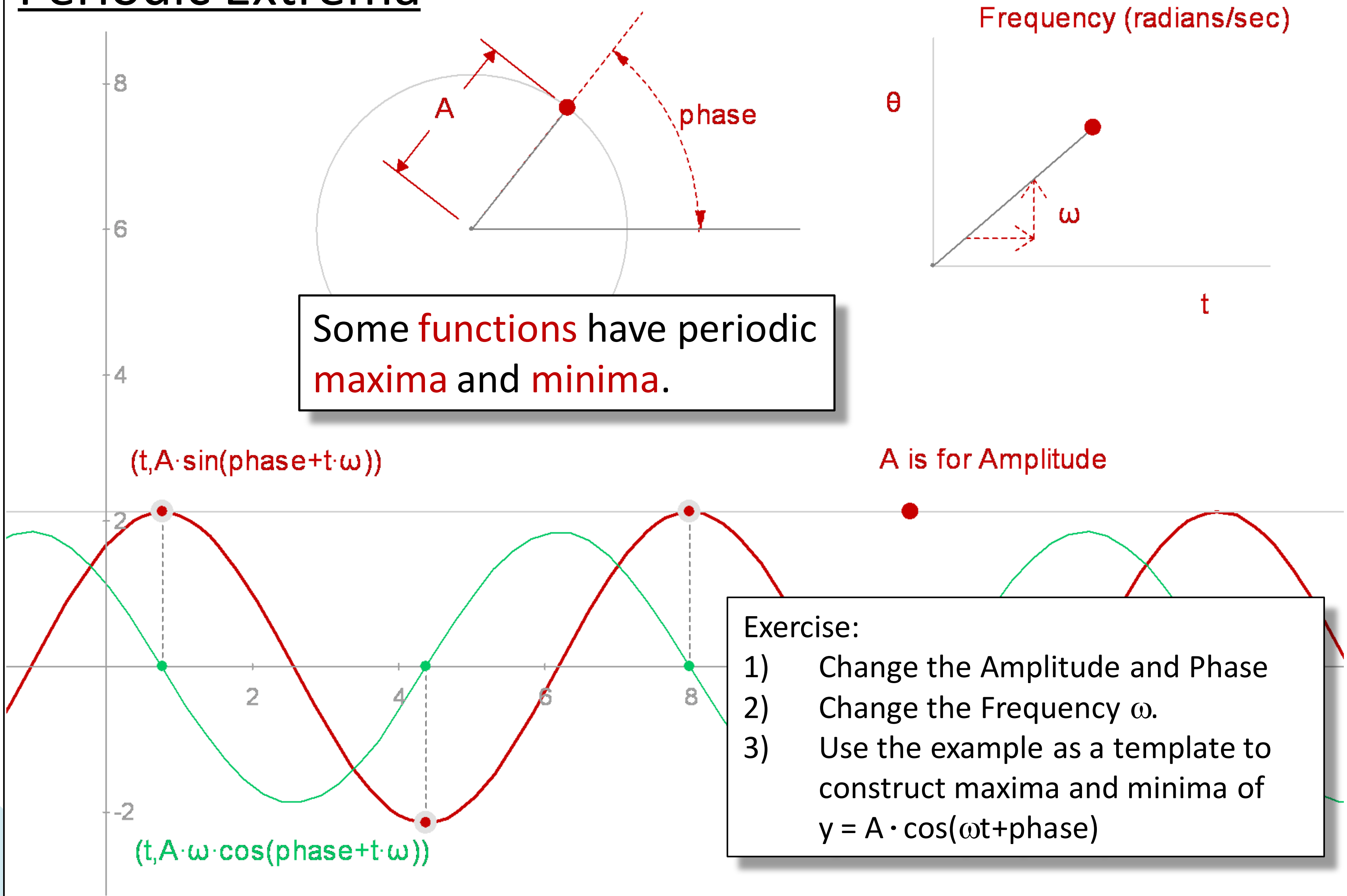
$$d+\frac{\left(\sqrt{b^2-3\,a\,c}-b\right)^3}{27\,a^2}+\frac{b\left(\sqrt{b^2-3\,a\,c}-b\right)^2}{9\,a^2}+\frac{c\left(\sqrt{b^2-3\,a\,c}-b\right)}{3\,a}$$

1) Write f(x).
2) Differentiate f(x).
3) Set f'(x) = 0. Solve for x.
4) Back substitute to find extremal values.

We use the Lisp functions `first()` *&* `second()` *to pick items out of the list Maxima returns.*

Periodic Extrema

Lecture16-FindingMaxMinPeriodic.gx



Some functions have periodic maxima and minima.

Exercise:

1) Change the Amplitude and Phase
2) Change the Frequency ω.
3) Use the example as a template to construct maxima and minima of $y = A \cdot \cos(\omega t + \text{phase})$

Periodic Extrema

Lecture16-FindingMaxMinPeriodic.wxm

```
(%i1) theta(t):=A*sin(t*w+phase);
(%o1) theta(t):=A sin(t w+phase)
```

1) Write f(x).

```
(%i2) diff(theta(t),t);
(%o2) w cos(t w+phase) A
```

2) Differentiate f(x).

```
(%i3) solve(%,t);
`solve' is using arc-trig functions
Some solutions will be lost.
```

$$(\%o3)\quad [\, t = -\frac{2\,phase - \%pi}{2\,w} \,]$$

3) Set f'(x) = 0. Solve for x.

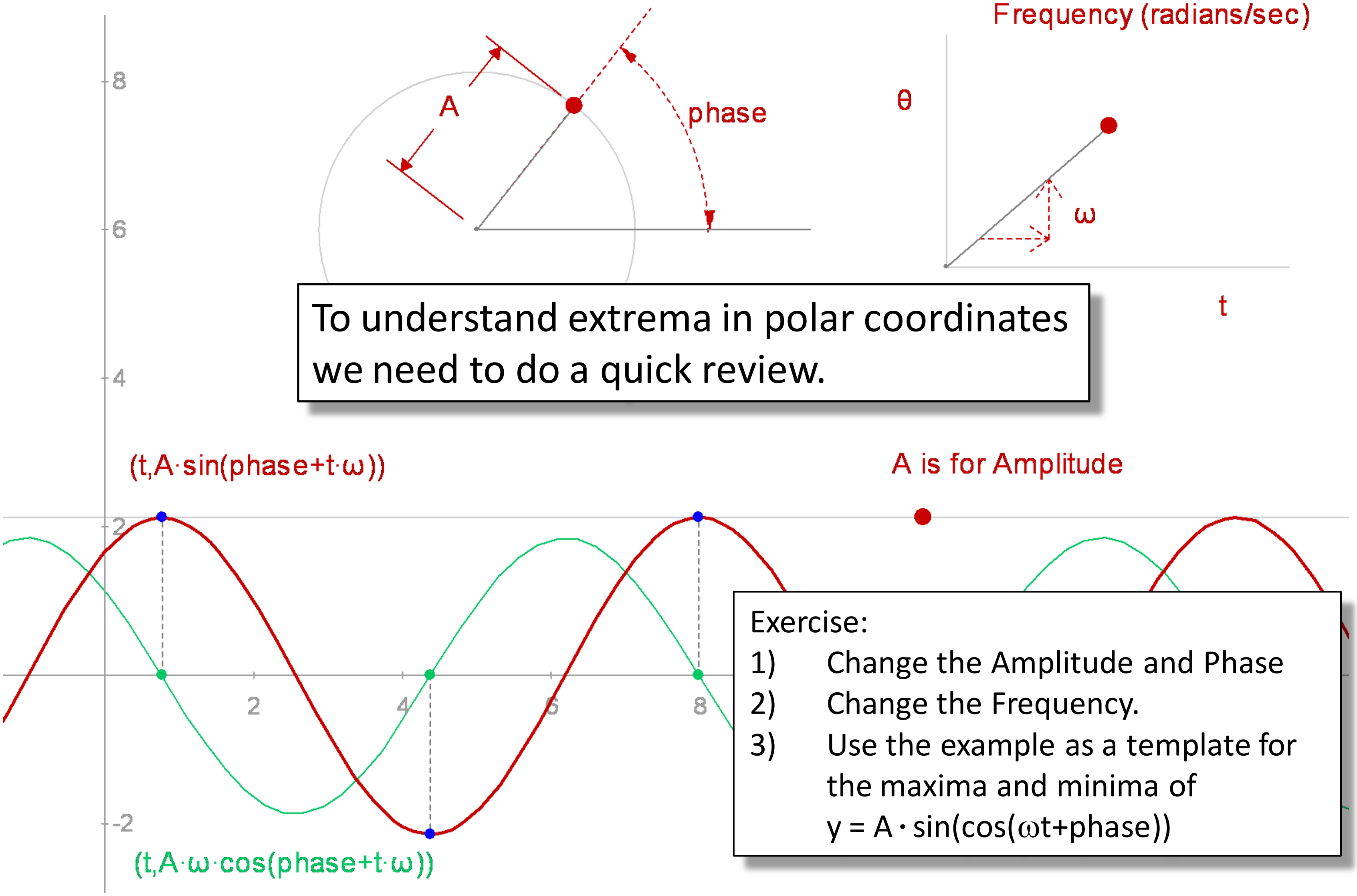
We use fact that f(x) is periodic to recover lost solutions.

```
(%i4) theta(second(first(%)));
(%o4) A
```

4) Back substitute to find extremal values.

Quick Polar Review

Lecture16-FindingMaxMinPeriodic.gx



To understand extrema in polar coordinates we need to do a quick review.

Exercise:

1) Change the Amplitude and Phase
2) Change the Frequency.
3) Use the example as a template for the maxima and minima of $y = A \cdot \sin(\cos(\omega t + phase))$

Extrema in Polar Coordinates

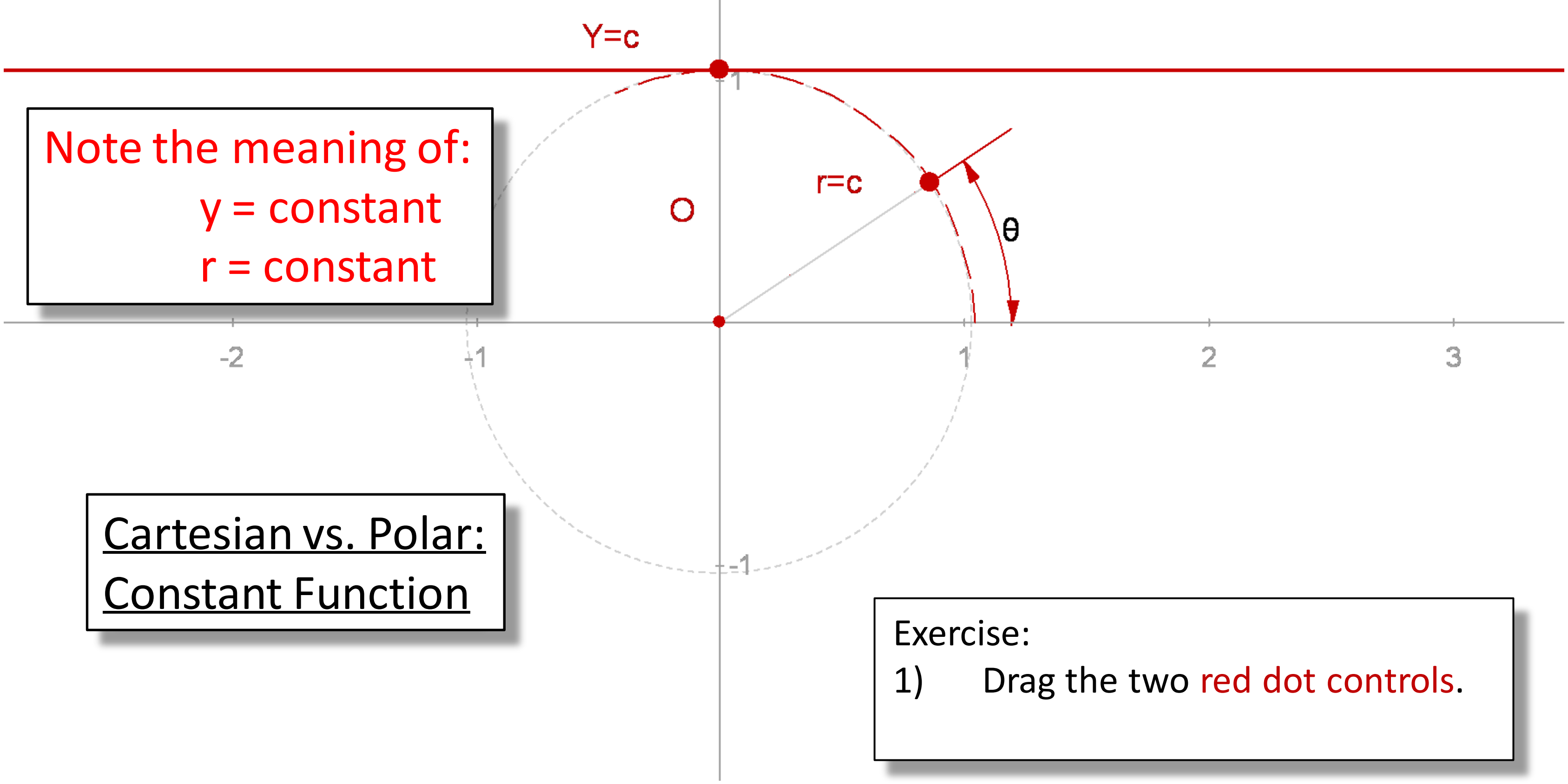
Lecture16-CartesianAndPolarConstant.gx

To understand extrema in polar coordinates we need to do a quick review.

Note the meaning of:

y = constant

r = constant



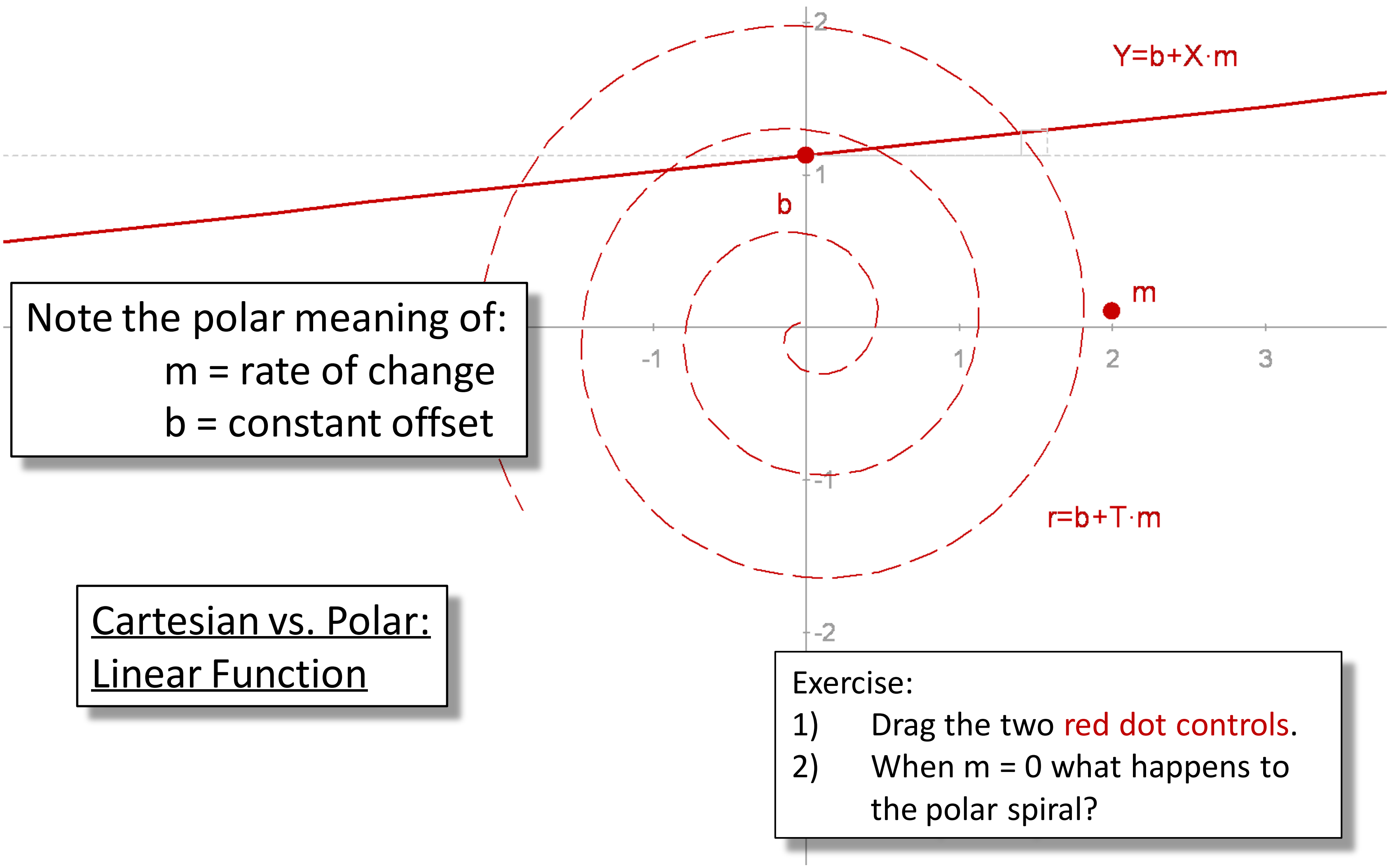
Cartesian vs. Polar:
Constant Function

Exercise:

1) Drag the two red dot controls.

Lecture16-CartesianAndPolarLinear.gx

Polar Extrema



Note the polar meaning of:

m = rate of change

b = constant offset

Cartesian vs. Polar: Linear Function

Exercise:

1) Drag the two red dot controls.
2) When m = 0 what happens to the polar spiral?

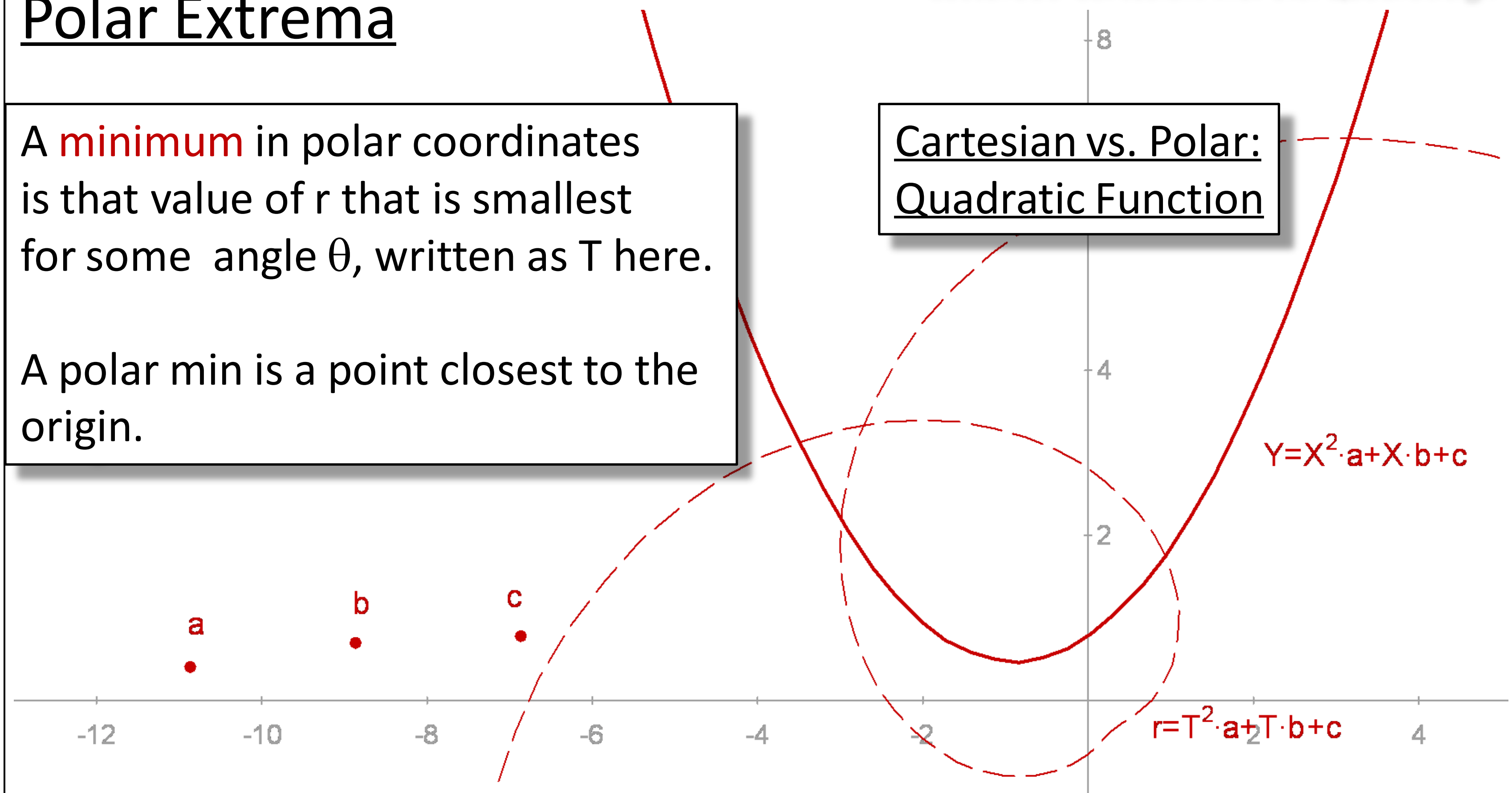
Polar Extrema

Lecture16-CartesianAndPolarQuadratic.gx

A minimum in polar coordinates is that value of r that is smallest for some angle θ, written as T here.

A polar min is a point closest to the origin.

Cartesian vs. Polar: Quadratic Function



Symbolically:

if ($r(k) \le r(T)$) for T in (a..b)

then $r(T)$ has a min at T=k

Exercise:

1) Drag the three red dot controls.
2) When a = 0 what happens to the polar curve?

Polar Extrema: Identical Procedure

Lecture16-FindingMaxMinPolar.wxm

```
r(t):= a*t^2+b*t+c;
```

$$r(t):=a\,t^2+b\,t+c$$

1) Write r(t).

```
diff(r(t),t);
```

$$2\,a\,t+b$$

2) Differentiate r(t).

```
solve(%,t);
```

$$[\,t=-\frac{b}{2\,a}\,]$$

3) Set r'(t) = 0. Solve for t.

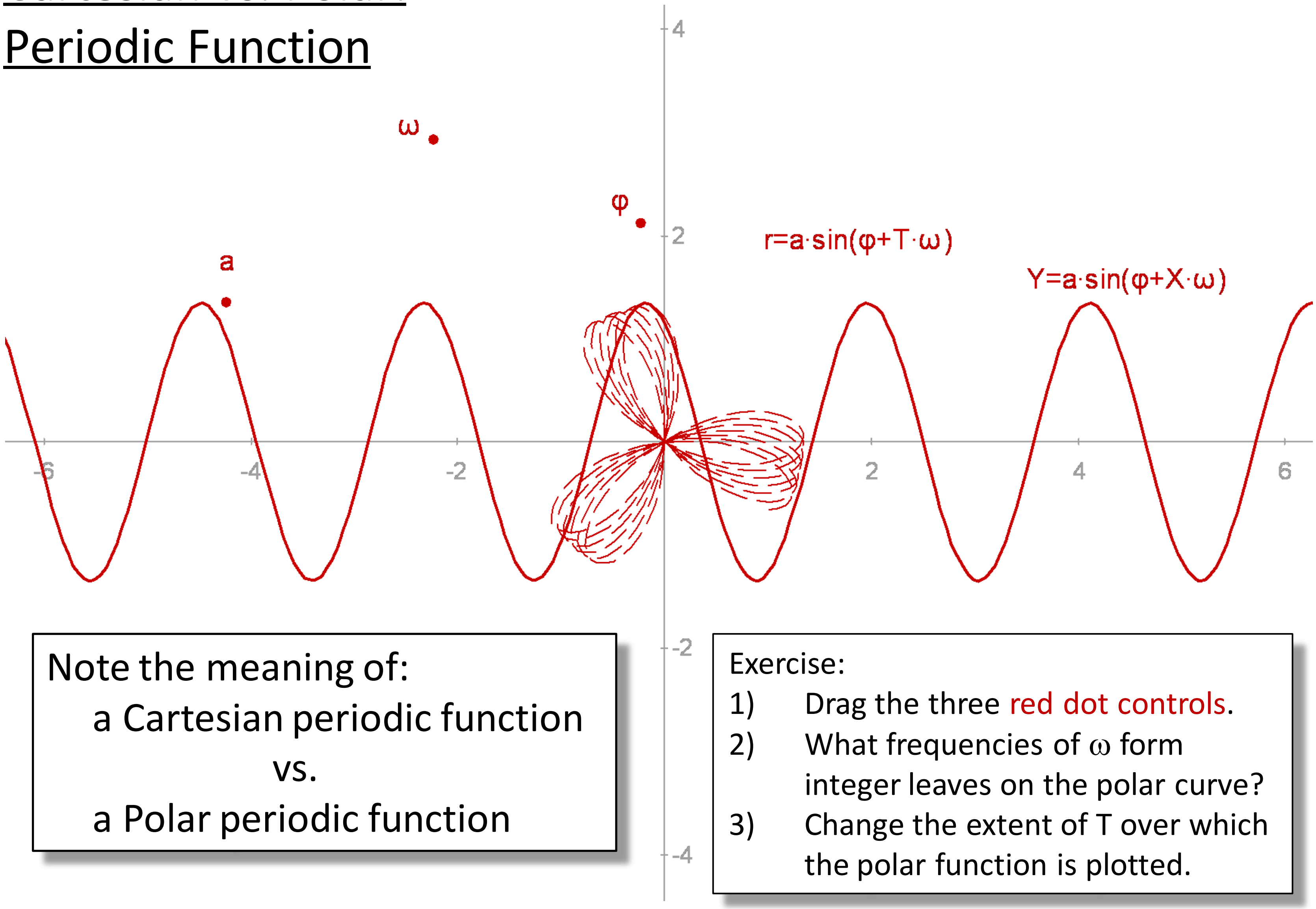
```
r(second(first(%)));
```

$$c-\frac{b^2}{4\,a}$$

4) Back substitute to find extremal values.

Cartesian vs. Polar: Periodic Function

Lecture16-CartesianAndPolarSinusoid.gx



Note the meaning of:

a Cartesian periodic function

vs.

a Polar periodic function

Exercise:

1) Drag the three red dot controls.
2) What frequencies of ω form integer leaves on the polar curve?
3) Change the extent of T over which the polar function is plotted.

Polar Extrema:

To find extrema of polar equations we note that the form of cartesian and polar equations are similar, but the shapes of the curve are different.

i) For the Cartesian min we look for points closest to the x-axis.

ii) For the Polar min we look for points closest to the origin.

This gives us a sense of how Cartesian points map to Polar and vica versa.

Since the curve may wind round and round, we have to trace the curve for a wide range of angle θ. Otherwise we may miss the closest or furthest point.

Exercise:
1) State the principle of statements i) and ii) above for the maxima.

Polar Extrema:

In the Cartesian case:
When we differentiate and set to zero we are finding the places where the rate of change of y with respect to x is momentarily zero.

In the Polar case:
When we differentiate and set to zero we are finding the places where the rate of change of distance r with respect to angle θ is momentarily zero.

In the Cartesian case:
We are looking for the place where the slope of the tangent line is zero, where the curve has approached and departed a horizontal asymptote.

In the Polar case:
The horizontal asymptotes are circles of r = constant!

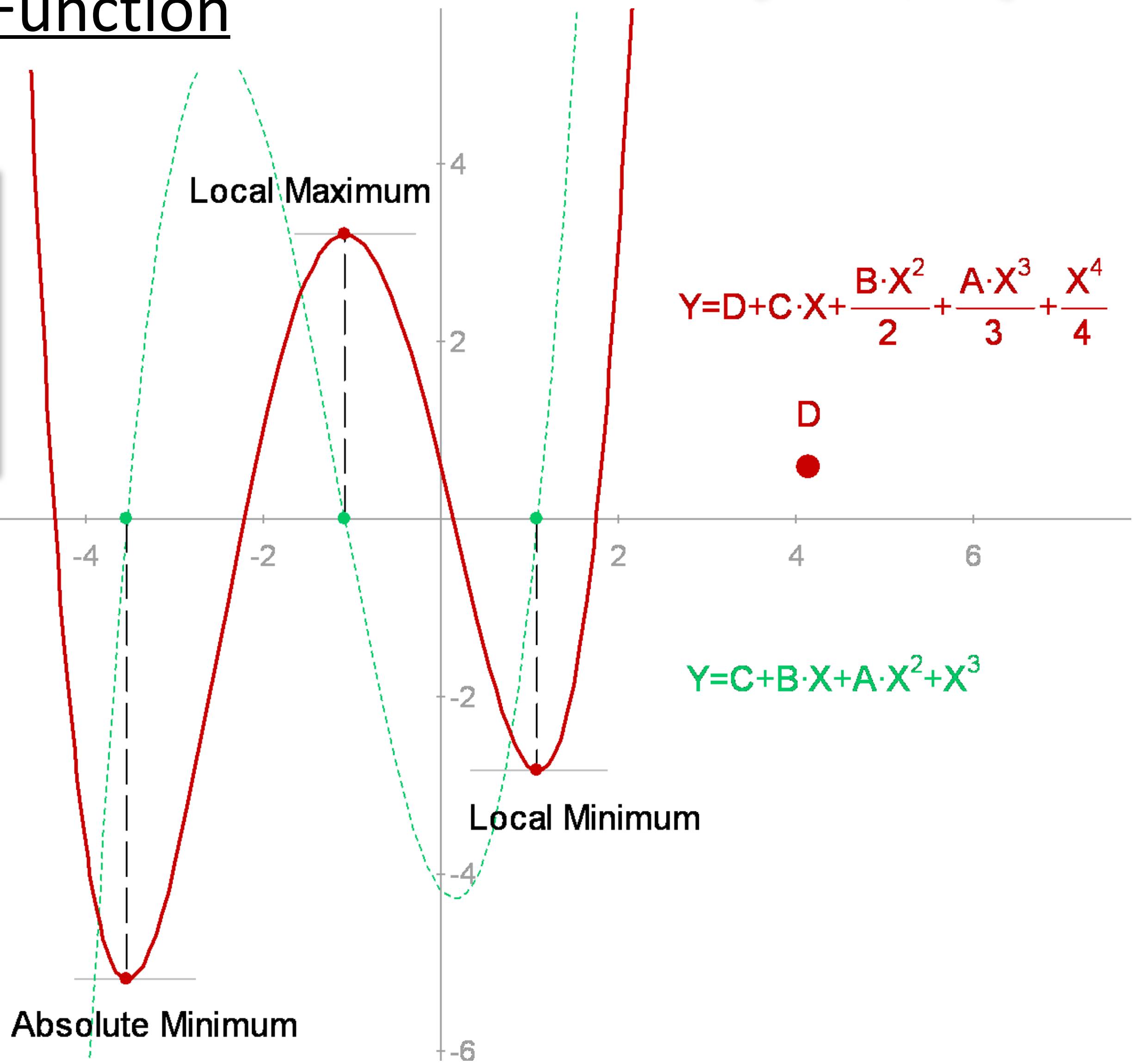
The symbolic procedure is the same as before, as the following shows:

Anatomy of a Function

Lecture16-FindingMaxMinQuartic.gx

Exercise:

1) Drag point D.
2) Does the derivative change?



Critical Points

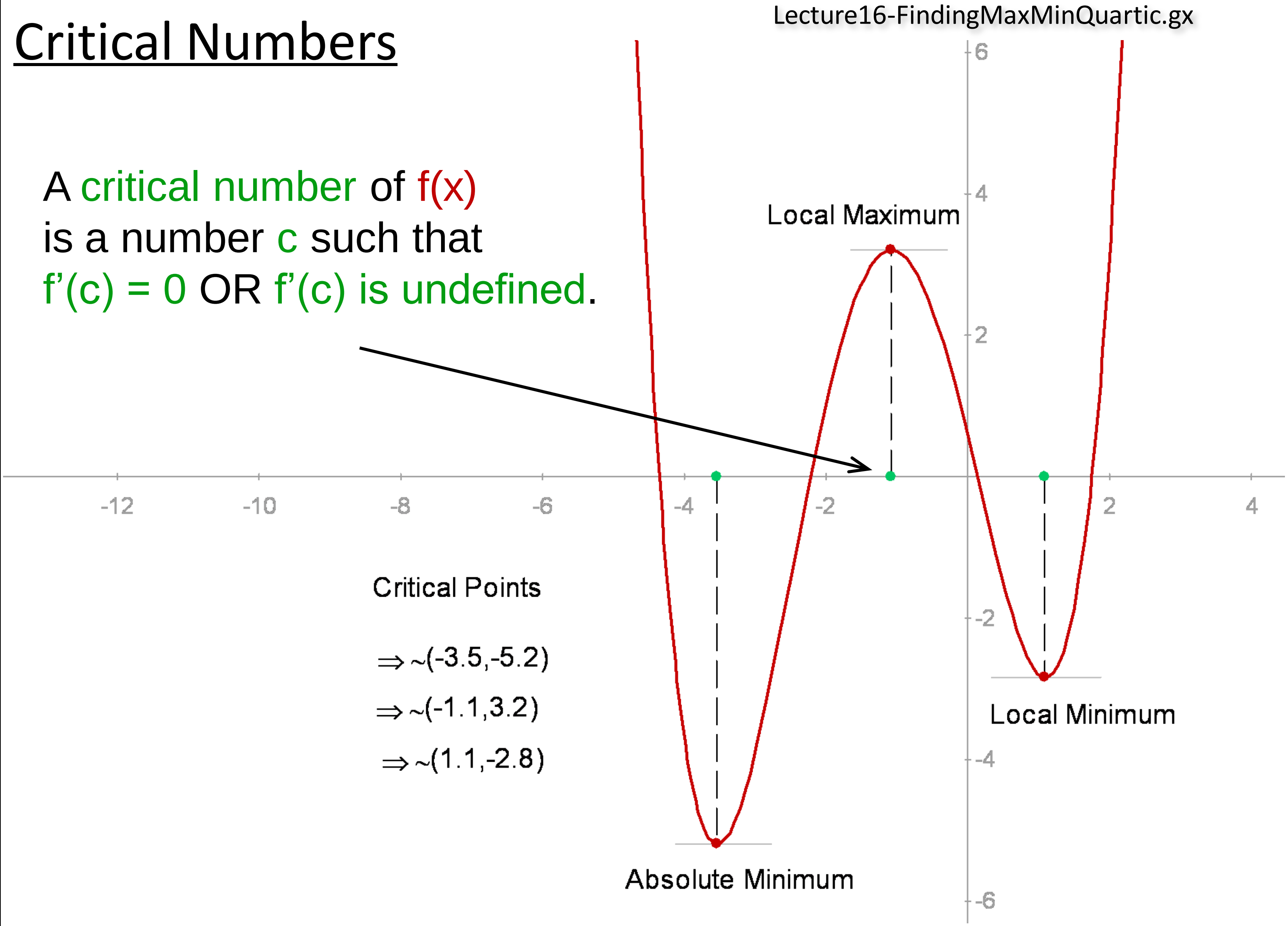
$\Rightarrow$ ~(-3.5,-5.2)

$\Rightarrow$ ~(-1.1,3.2)

$\Rightarrow$ ~(1.1,-2.8)

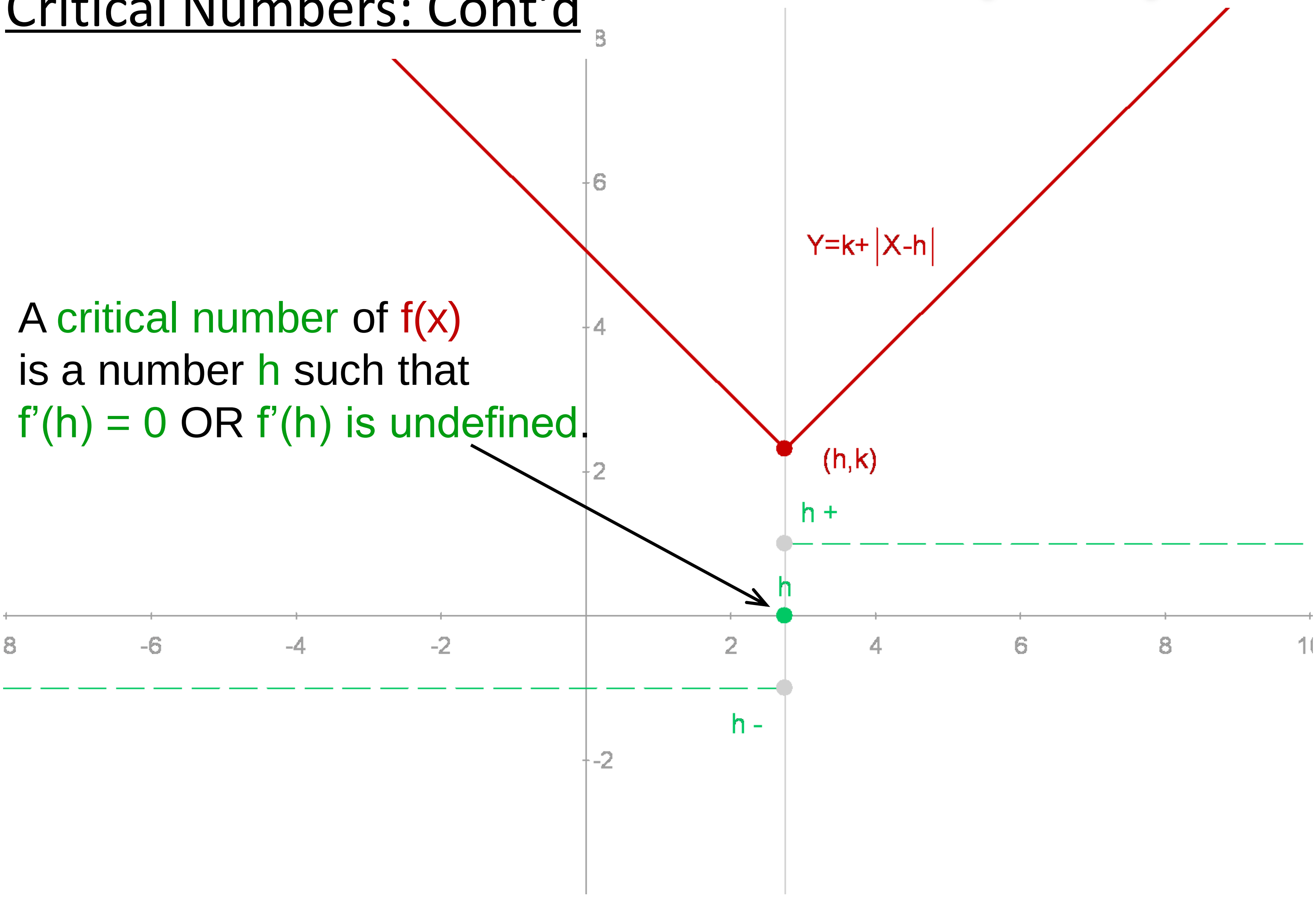
Critical Numbers

A critical number of f(x)
is a number c such that
$f'(c) = 0$ OR $f'(c)$ is undefined.



Critical Numbers: Cont'd

Lecture16-FindingMaxMinAbs.gx

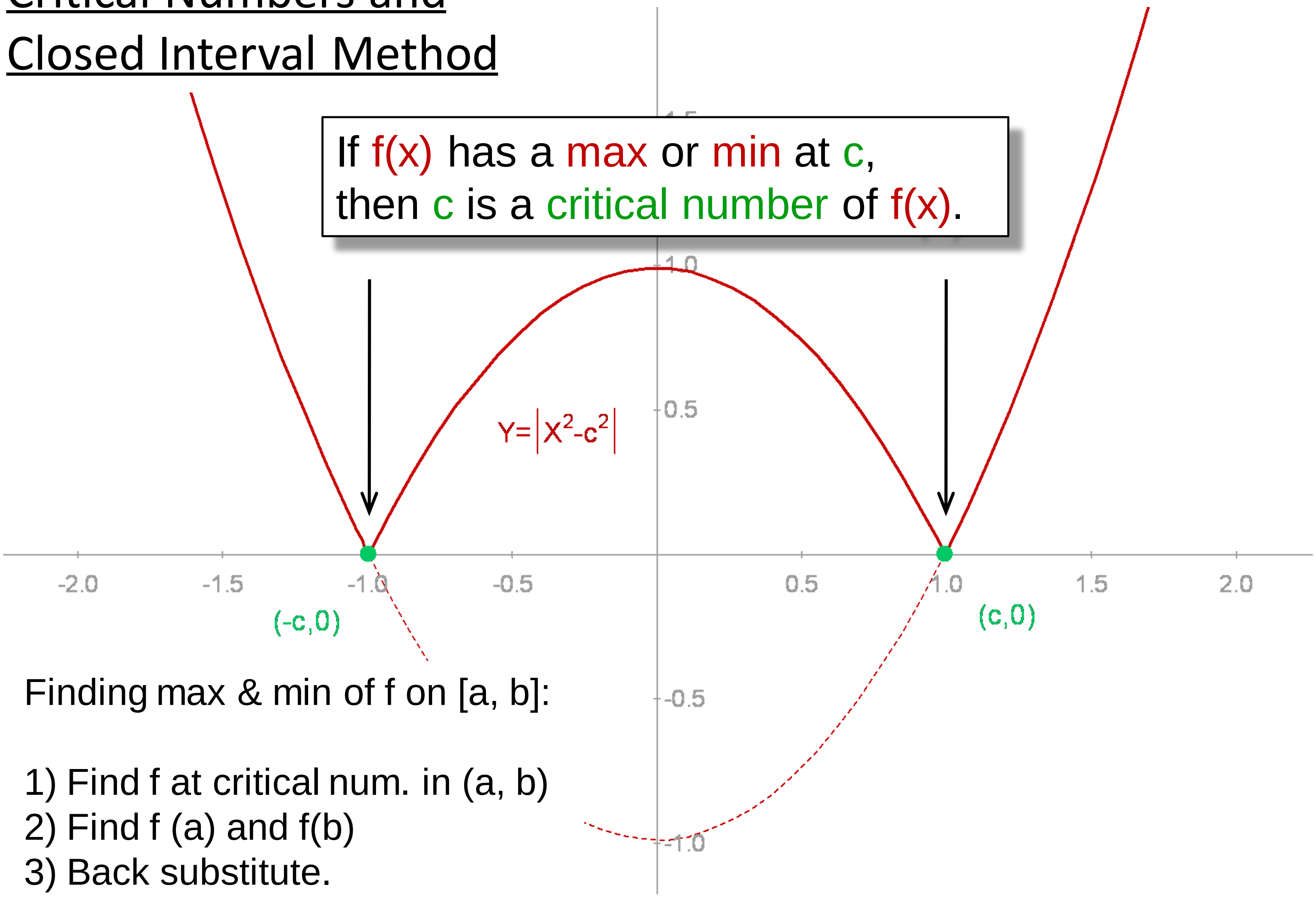


A critical number of f(x)
is a number h such that
f'(h) = 0 OR f'(h) is undefined.

Critical Numbers and Closed Interval Method

Lecture16-FindingMaxMinAbsQuad.gx

If f(x) has a max or min at c, then c is a critical number of f(x).

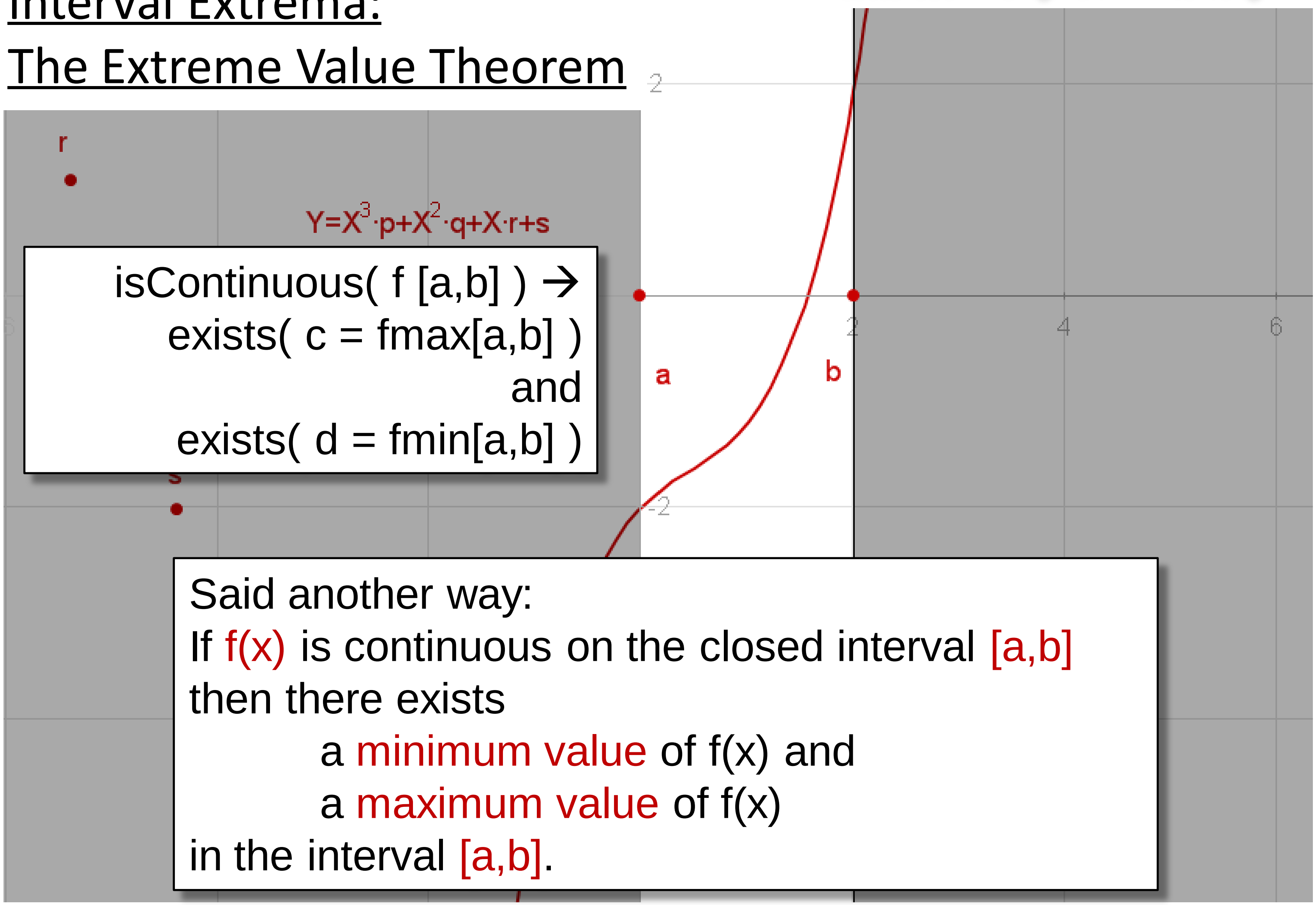


Finding max & min of f on [a, b]:

1) Find f at critical num. in (a, b)
2) Find f (a) and f(b)
3) Back substitute.

Interval Extrema:
The Extreme Value Theorem

Lecture16-FindingMaxMinInterval.gx



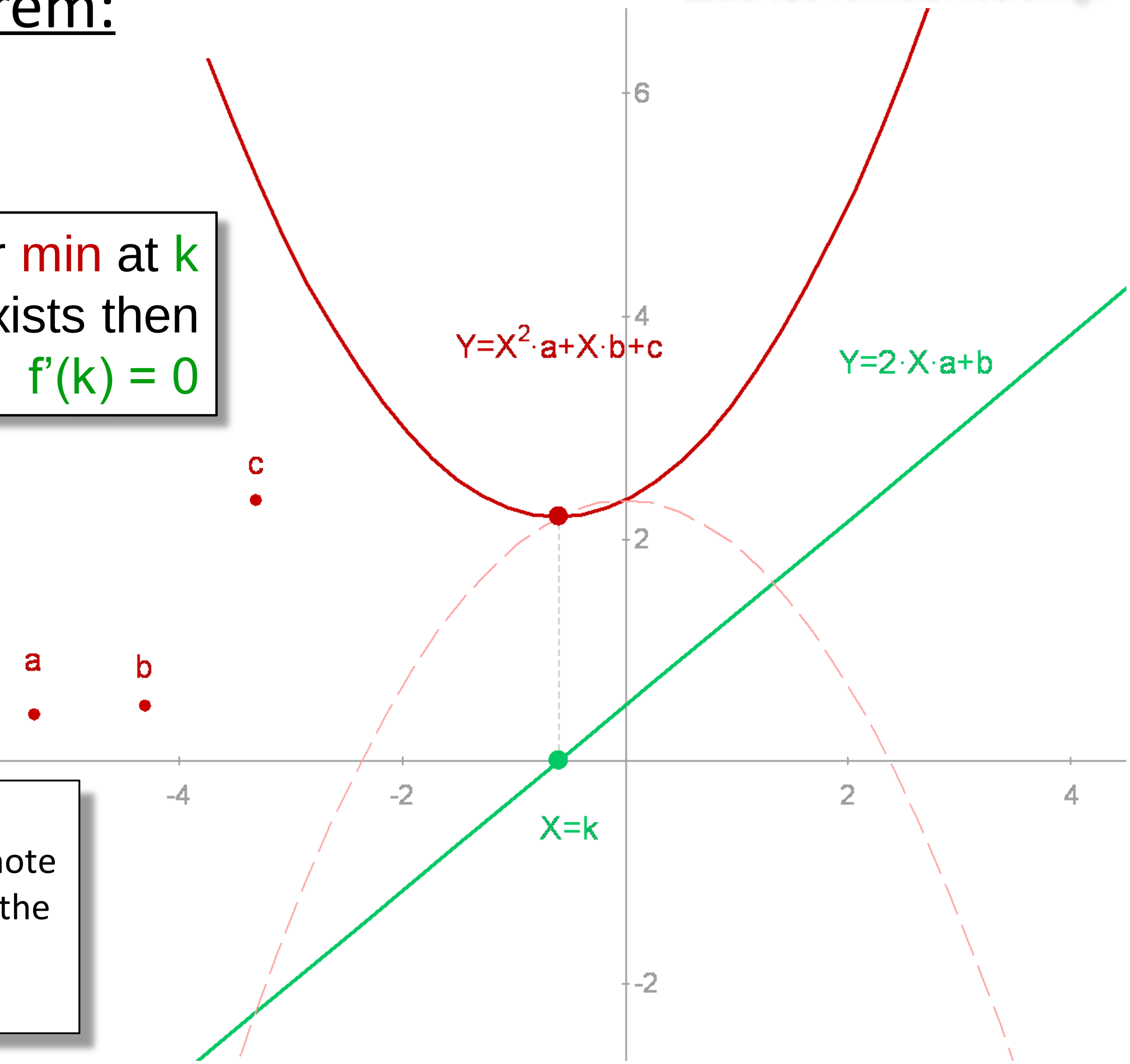
isContinuous(f [a,b]) → exists(c = fmax[a,b]) and exists(d = fmin[a,b])

Said another way:
If f(x) is continuous on the closed interval [a,b] then there exists
 a minimum value of f(x) and
 a maximum value of f(x)
in the interval [a,b].

Fermat's Theorem:

Lecture16-FermatsTheorem.gx

if f has a max or min at k and f'(k) exists then f'(k) = 0



Exercise:

1) Drag b and note the locus of the minimum.

Lecture16-FermatsTheorem.wxm

The Take Home Message:

```
y(X):=a*X^2+b*X+c;
```

$$y(X):=a\,X^2+b\,X+c$$

1) Write y(X).

```
yp(X):=diff(y(X),X);
```

$$yp(X):=diff(y(X),X)$$

2) Differentiate y(X).

```
solutions:solve(yp(X),X);
```

$$[X=-\frac{b}{2\,a}]$$

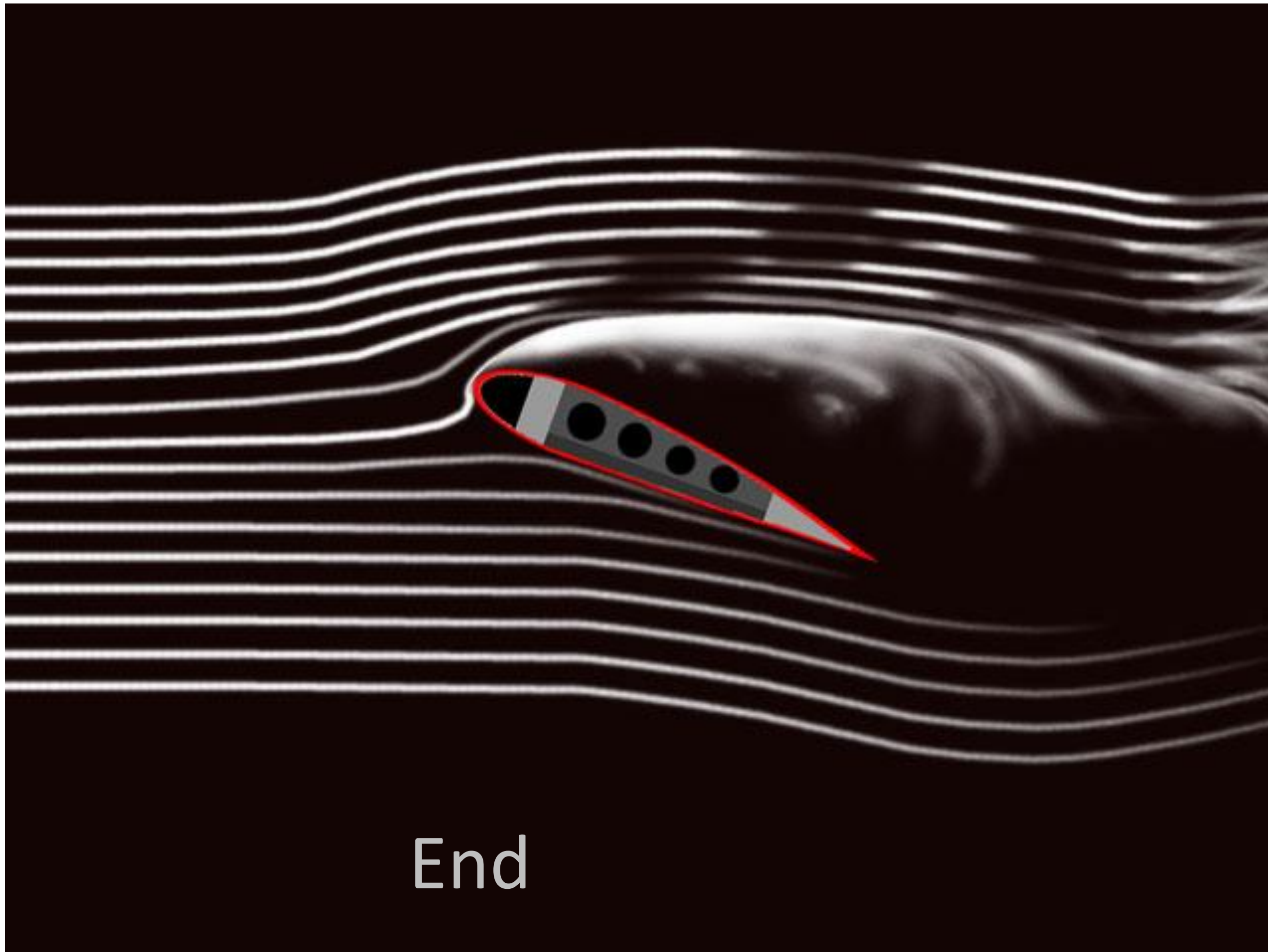
3) Find zeros of yp(X).

```
y(second(first(solutions)));
```

$$c-\frac{b^2}{4\,a}$$

4) Back substitute to find extrema.

NOTE: "x vs. X"
Geometry Expressions™ and wxMaxima are CASE SENSITIVE!


End