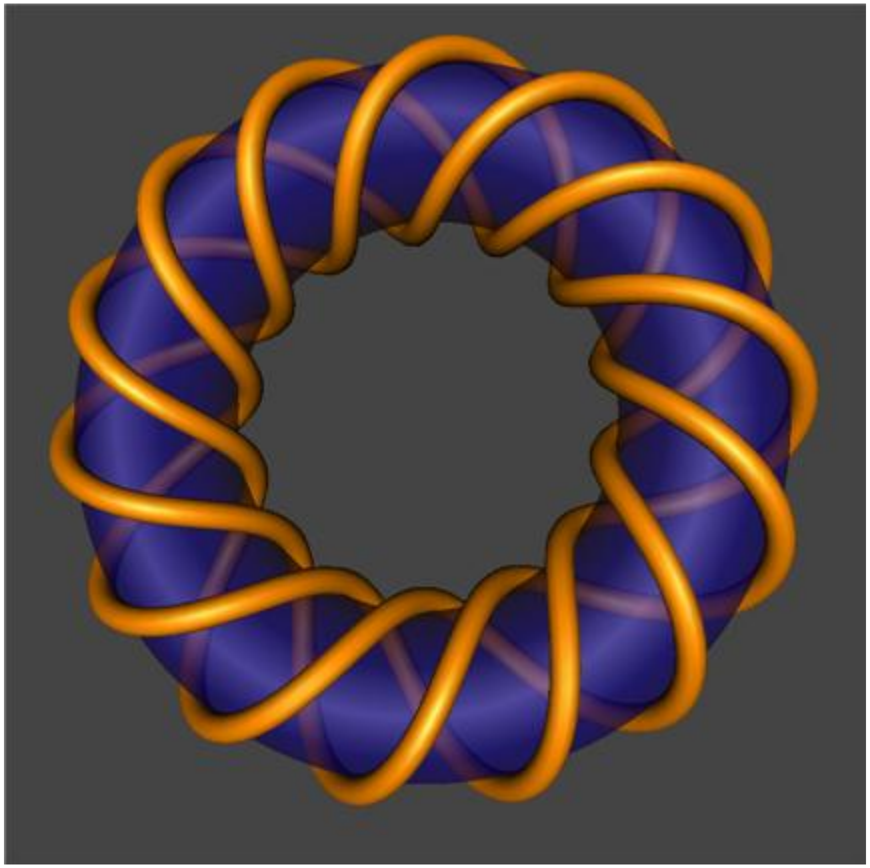
Learning Calculus With Geometry Expressions™

by L. Van Warren



- Torus Knot(3,14) using KnotPlot™

Lecture 18: Optimization

Chapter 4: Rates and Extremes

Lecture	Topic
14	Rates of Change
15	Related Rates
16	Extrema – Maxima and Minima
17	Derivative Tests and Mean Value Theorem
18	**Optimization**

Inspiration

- George Bergman

William Thurston, Ph.D 1946 -
Princeton Mathematician

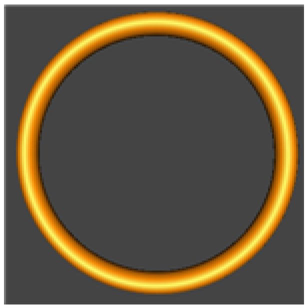
Pioneer in Low-Dimensional Topology
Geometrization "Monster" Theorem
Laid Foundation for Proof of Poincare Conjecture

Notable Awards: Fields Medal (1982)
Oswald Veblen Prize (1976)

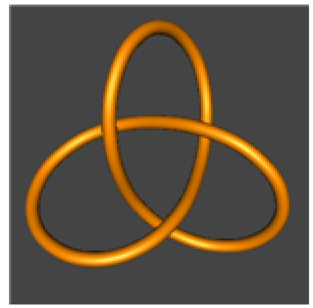
Worked on: Foliation Theory
Knot Theory
Hyperbolic 3-Manifolds
Not Knot Movie

Read: On Proof and Progress in Mathematics

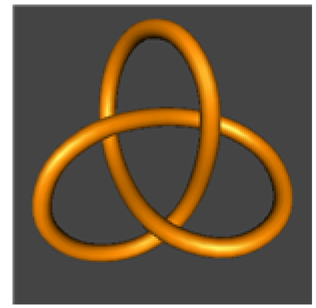
Selected Knots:



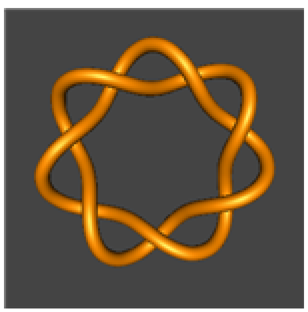
0_1 - The UnKnot



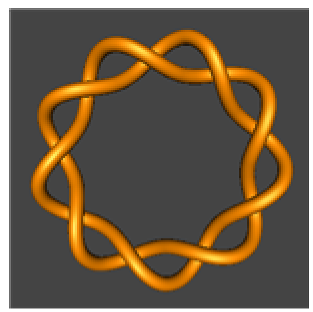
3_1 - The Trefoil



3_2 - Mirrored Trefoil



7_1



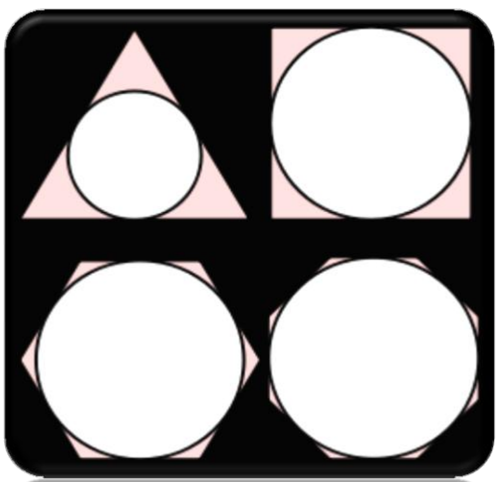
9_1

9_{23}

- Knots rendered using KnotPlot™ by [Robert G. Scharein]

<u>Single Variable Optimization</u>

Works if there is a single variable that survives differentiation.

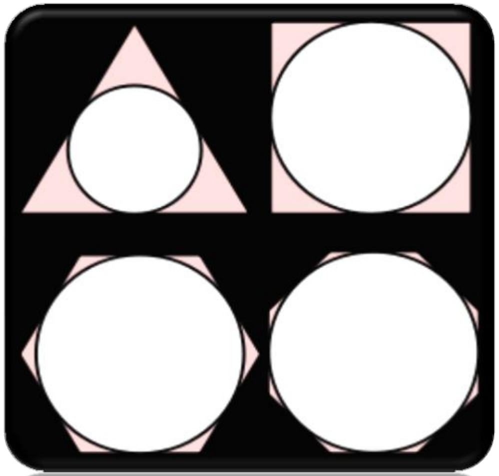


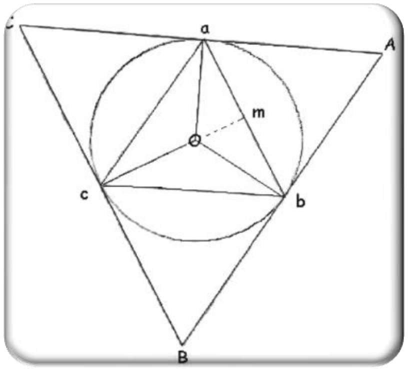
Steps:

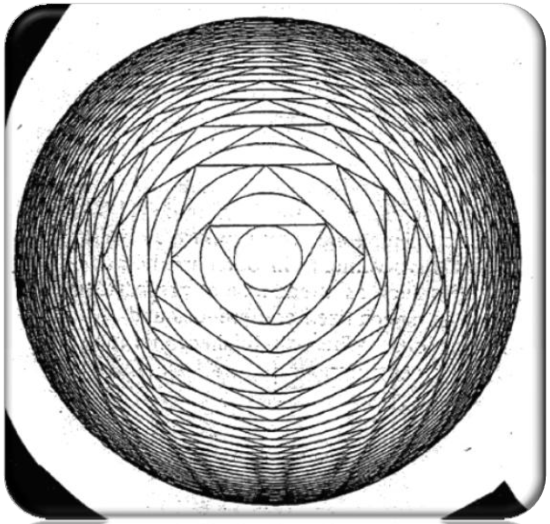
1) Draw problem
2) Write formulas
3) Eliminate variables
4) Set Derivative = 0
5) Solve
6) Verify

Shape in Shape Problems: Constrained Optimization

Largest x in y where {x, y} = {polygon, circle, curve}



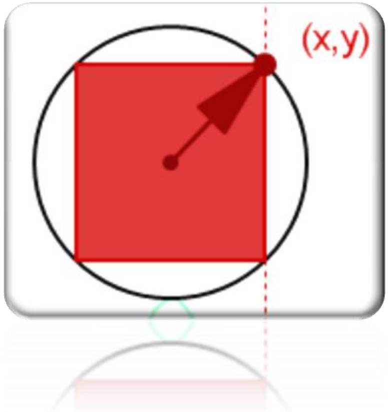




Largest Rectangle In Circle:

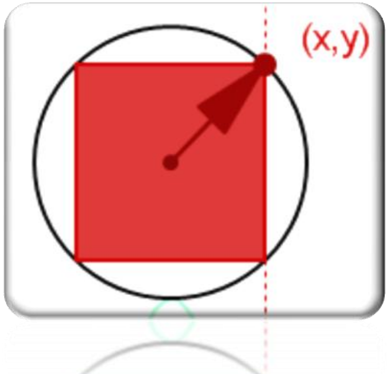
Steps:

1) Draw problem
2) Write formulas
3) Reduce equation to one variable
4) Set Derivative = 0
5) Solve
6) Verify



Largest Rectangle In Circle:

What is true for one circle is true for another, modulo a scale factor, so we will use a unit circle of radius 1. Without loss of generality we can pretend that the circle and the square are centered at the origin. Points on the circle must then satisfy:



$$x^2 + y^2 = 1$$

The area of the inscribed rectangle is:

$$Area = 2x \cdot 2y = 4xy$$

The "trick" in single variable optimization is to get rid of all the variables but one. We solve the top equation for y and substitute it into the equation for area to obtain an expression in x alone. We then differentiate this expression and find the critical point. (Note that we didn't need the area of the circle!)

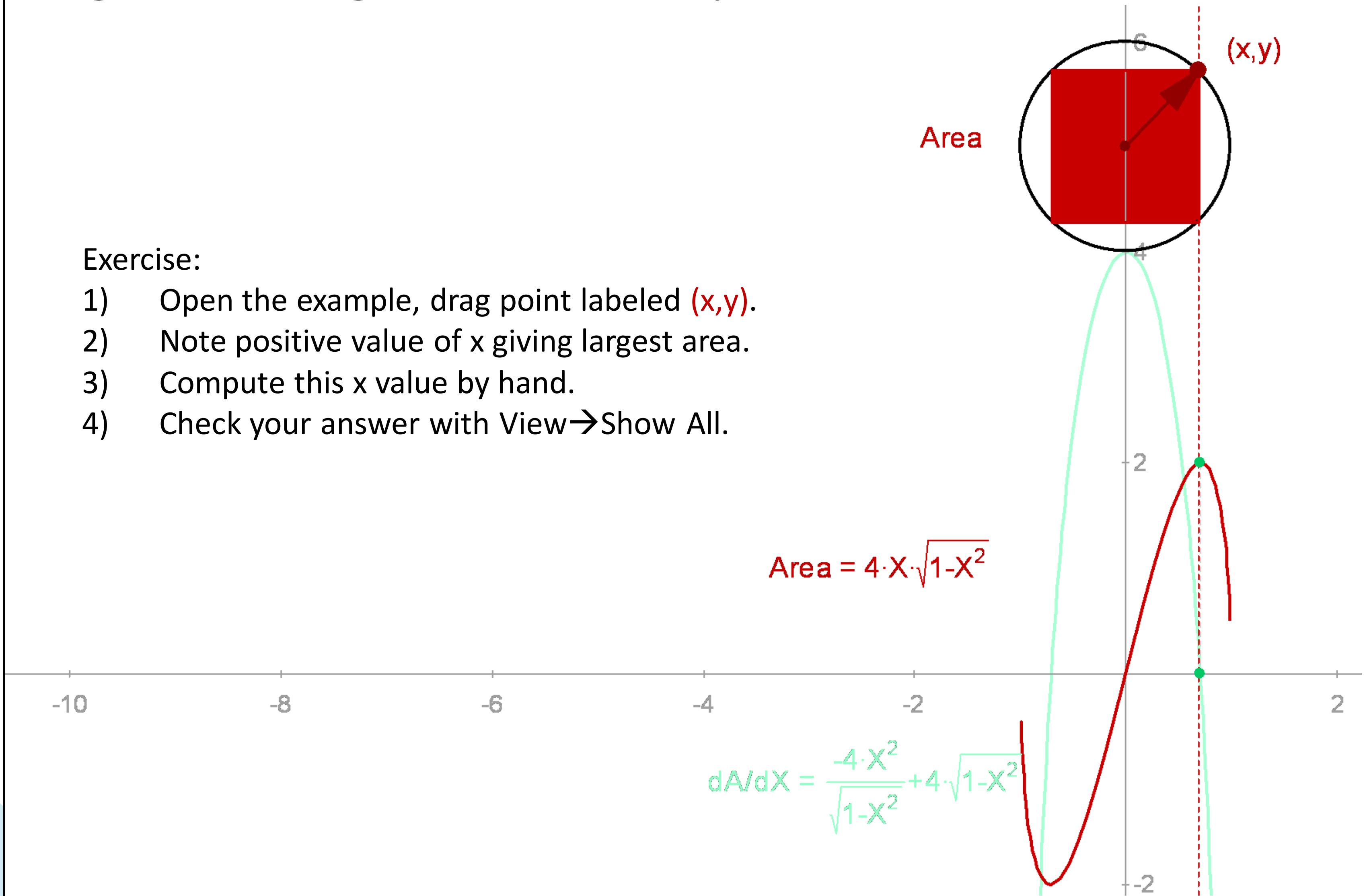
Exercise: Do this calculation using the product rule.

Lecture18-SquareInCircle.gx

Largest Rectangle In Circle Is Square:

Exercise:

1) Open the example, drag point labeled (x,y).
2) Note positive value of x giving largest area.
3) Compute this x value by hand.
4) Check your answer with View→Show All.



Largest Rectangle In Circle: Equation

Lecture18-SquareInCircle.wxm

```
circle: x^2+y^2=1;
```
$y^2 + x^2 = 1$

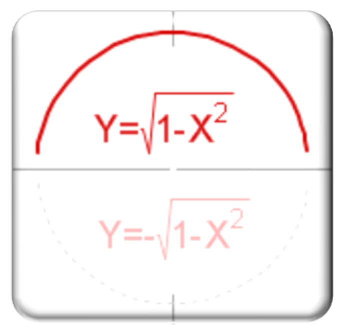
```
bothHalves : solve(circle,y);
```
$[y = -\sqrt{1 - x^2}, y = \sqrt{1 - x^2}]$

```
topHalf : second(bothHalves);
```
$y = \sqrt{1 - x^2}$

1) Write equation for circle.
2) Solve equation for y. Two 'halves" result!
3) We will work with the top half.

`second()` is a list operator, fetching the second solution.

`first()` through `tenth()` also work to extract elements of a list.



Largest Rectangle In Circle: Size

Lecture18-SquareInCircle.wxm

```
height(x):= 2*rhs(''topHalf);
```

$$height(x):=2\,rhs\left(y=\sqrt{1-x^2}\right)$$

```
height(x);
```

$$2\sqrt{1-x^2}$$

```
height(u);
```

$$2\sqrt{1-u^2}$$

```
width(x) := 2*x;
```

$$width(x):=2\,x$$

1) The height of the inscribed rectangle is twice the y value.

   The two-single quotes tell maxima to evaluate the argument right now and rhs() is maxima shorthand for right-hand side.

   Otherwise functions don't evaluate their definitions until the last possible moment, a policy called "lazy evaluation".

2) We call height(u) as a quality control measure to check that the function definition is correct.

3) Then we compute the width, but it requires no "tricks".

Largest Rectangle In Circle: Area

Lecture18-SquareInCircle.wxm

```
Area(x):=height(x)*width(x)$
```

```
Area(x);
```

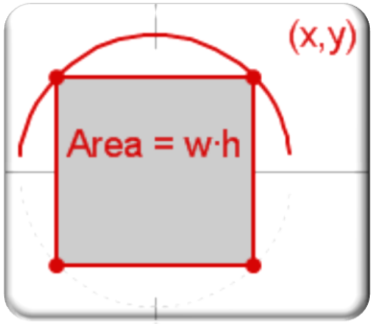
$4\,x\sqrt{1-x^2}$

```
Area(u);
```

$4\,u\sqrt{1-u^2}$

1) This function calls two other functions to compute the area of the rectangle.
2) We invoke the function again so we can see its definition.
3) We then do some quality control to make sure the definition is general and the variables are bound properly.

`u` is a dummy variable. We could have used any name here. If our definitions were faulty this test could reveal that.



Largest Rectangle In Circle: Derivative

Lecture18-SquareInCircle.wxm

```
Area(u);
```

$$4\,u\sqrt{1-u^2}$$

```
dAdx : diff(Area(x),x);
```

$$4\sqrt{1-x^2}-\frac{4\,x^2}{\sqrt{1-x^2}}$$

```
roots : solve(dAdx,x);
```

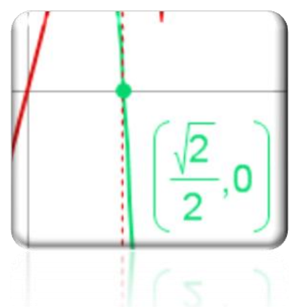
$$[\,x=-\frac{1}{\sqrt{2}},\ x=\frac{1}{\sqrt{2}}\,]$$

1) Quality Control Check

2) Differentiate the area with respect to x. This works because we eliminated y earlier.

3) Find the zero’s where the critical points live.

solve is a euphemism for finding the values of x at which the rate of change is zero.



Largest Rectangle In Circle: Derivative

Lecture18-SquareInCircle.wxm

```
roots : solve(dAdx,x);
```

$$[x=-\frac{1}{\sqrt{2}}, x=\frac{1}{\sqrt{2}}]$$

1) Roots computed.

```
Xcritical : rhs(second(roots));
```

$$\frac{1}{\sqrt{2}}$$

2) Find critical X.

```
Ycritical : Area(Xcritical );
```

$$2$$

3) Back substitute to find critical Y.

```
Pcritical : [Xcritical, Ycritical];
```

$$[\frac{1}{\sqrt{2}}, 2]$$

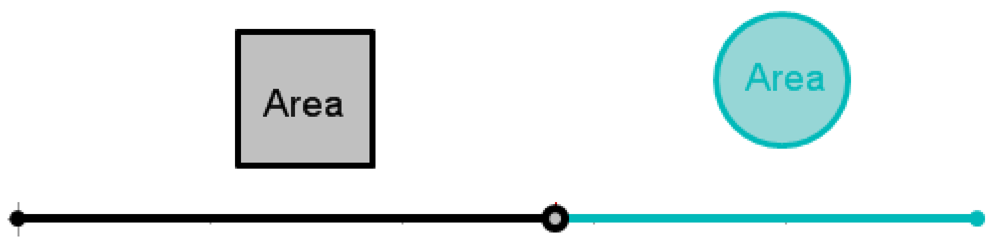
4) Put the critical x and y in a point, an ordered pair.

Exercise: Show how this proves the largest rectangle is a square.

The Wire Problem:

Description:

- A single piece of wire is cut in two lengths.
- One piece is formed into a circle.
- The other is formed into a square.
- Find the length that minimizes the area of both figures.
- Find the length that maximizes the area of the circle.
- Find the length that maximizes the area of the square.

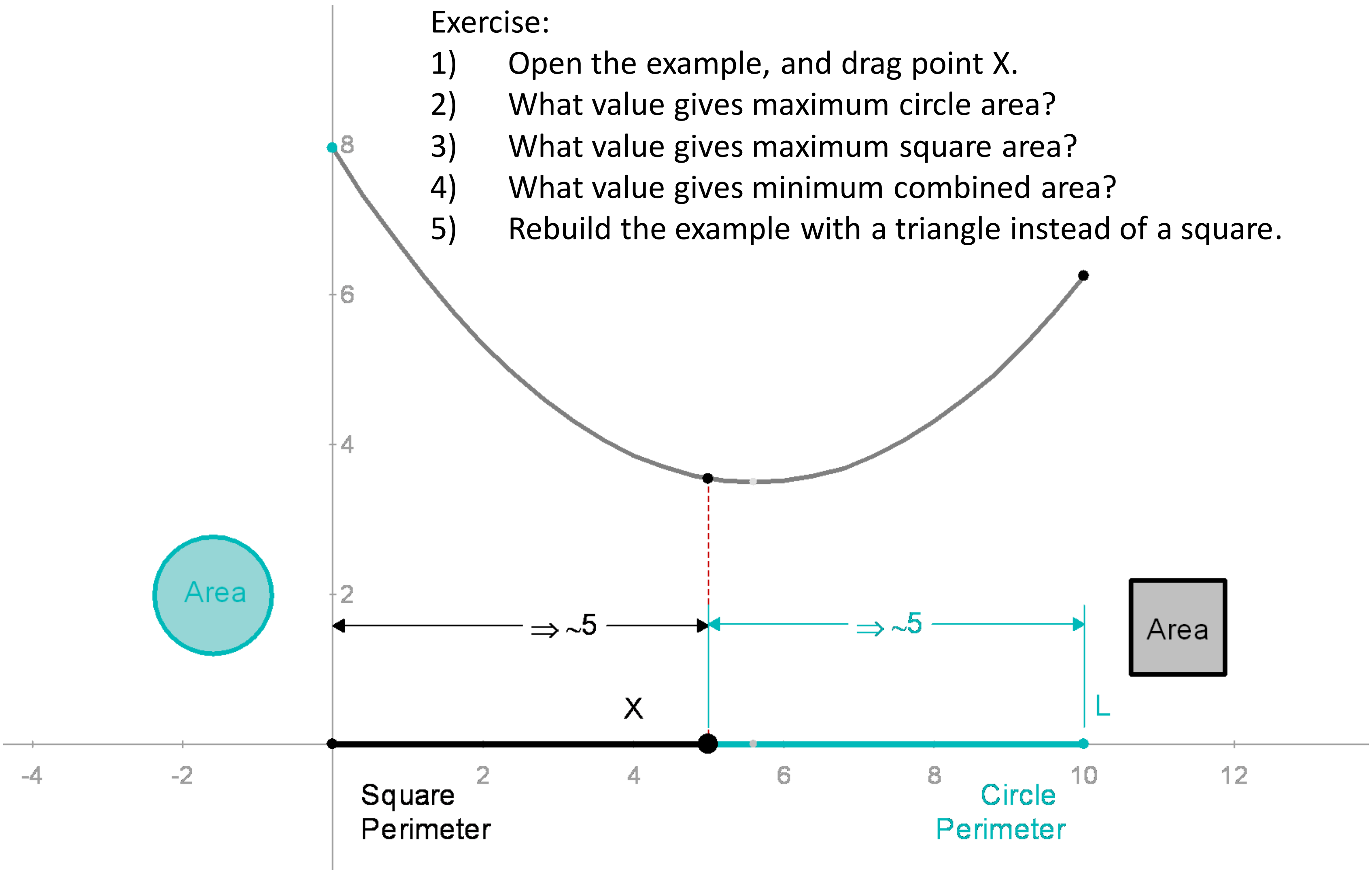


Lecture18-OptimalPerimeter.gx

The Wire Problem:

Exercise:

1) Open the example, and drag point X.
2) What value gives maximum circle area?
3) What value gives maximum square area?
4) What value gives minimum combined area?
5) Rebuild the example with a triangle instead of a square.



The Wire Problem

Lecture18-OptimalPerimeter.wxm

```
Asquare(x):=(x/4)^2;
```

$$\text{Asquare}(x) := \left(\frac{x}{4}\right)^2$$

1) Write area of square in terms of the perimeter x.


```
Acircle(L,x):=%pi*( (L-x)/(2*%pi) )^2;
```

$$\text{Acircle}(L, x) := \%pi\left(\frac{L - x}{2\,\%pi}\right)^2$$

2) Write area of circle in terms of the perimeter L-x where r = perimeter/2π.


```
Acircle(L,x);
```

$$\frac{(L - x)^2}{4\,\%pi}$$

3) Maxima simplifies.

`%pi` is a maxima symbol for π.

The Wire Problem

Lecture18-OptimalPerimeter.wxm

```
Atotal(L,x):= radcan(Asquare(x)+Acircle(L,x));
```

Atotal(L, x):=radcan(Asquare(x)+Acircle(L, x))

1) Write total area as sum of two functions.

```
Atotal(L,x);
```

$$\frac{4\,L^2-8\,x\,L+(\%pi+4)x^2}{16\,\%pi}$$

2) Is this the simplest form?

3) Differentiate the area with respect to x.

```
dAdx(L,x):=radcan(diff(Atotal(L,x),x));
```

dAdx(L, x):=radcan(diff(Atotal(L, x), x))

```
dAdx(L,x);
```

$$-\frac{4\,L+(-\%pi-4)x}{8\,\%pi}$$

`Radcan()` tells maxima to provide the simplest rational form. What is a rational form?

The Wire Problem

Lecture18-OptimalPerimeter.wxm

```
solve(dAdx(L,x),x);
```

$[\, x = \frac{4\,L}{\%pi + 4} \,]$

1) Find the x value at which the derivative is zero.

```
Atotal(L,rhs(first(%)));
```

$\frac{L^2}{4\,\%pi + 16}$

2) Back substitute to find the minimum (or maximum) total area.

Exercise:

1) How do we determine whether `Atotal` is a maximum or minimum?
2) Extend the calculation to prove the critical point as maximum or minimum.
3) In the `x` interval `[0,L]`, there are two critical points.
   What are they and what do they represent?

`rhs()` fetches the right hand side of an equivalence relation? What is that?

Asteroid Impact, A Dramatic Example:

- Image Courtesy NASA

"One can often solve a class of problems
for the same effort as solving a single problem."

- Will Worley

Asteroid Closest Approach:

A space mission to steer an asteroid away from the earth has failed. The electromagnetic pulse from multiple nuclear detonations has disabled the internet and many personal computers.

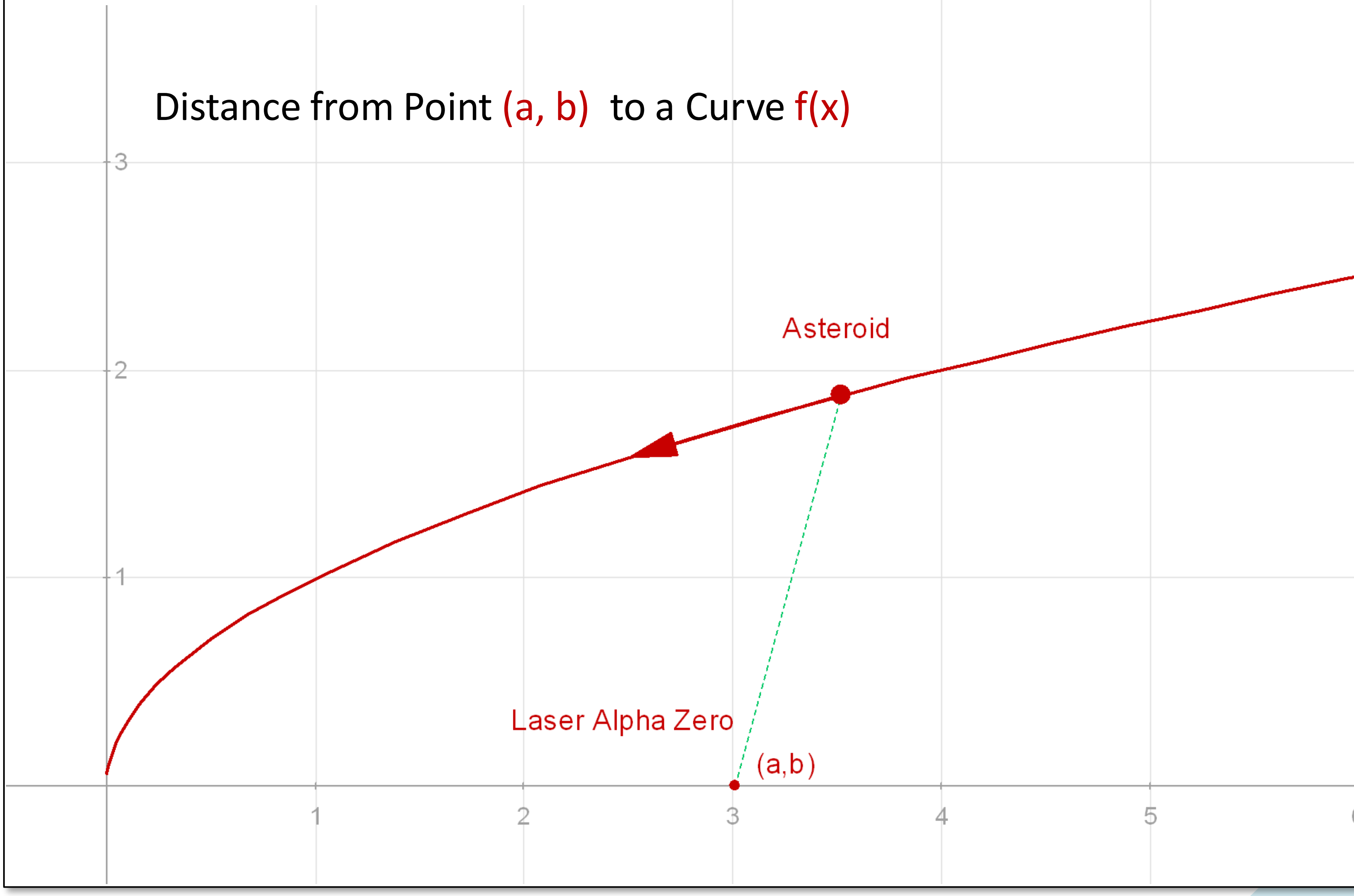
You are one of many Calculus classes recruited to compute the asteroid's closest position so that ground-based lasers can be used to heat its surface and finish deflecting it.

There will only be one chance to fire the lasers and the altitude of closest approach must be dictated in thirty minutes over a noisy ham radio connection.

Asteroid Closest Approach:

Lecture18-ClosestApproach.gx

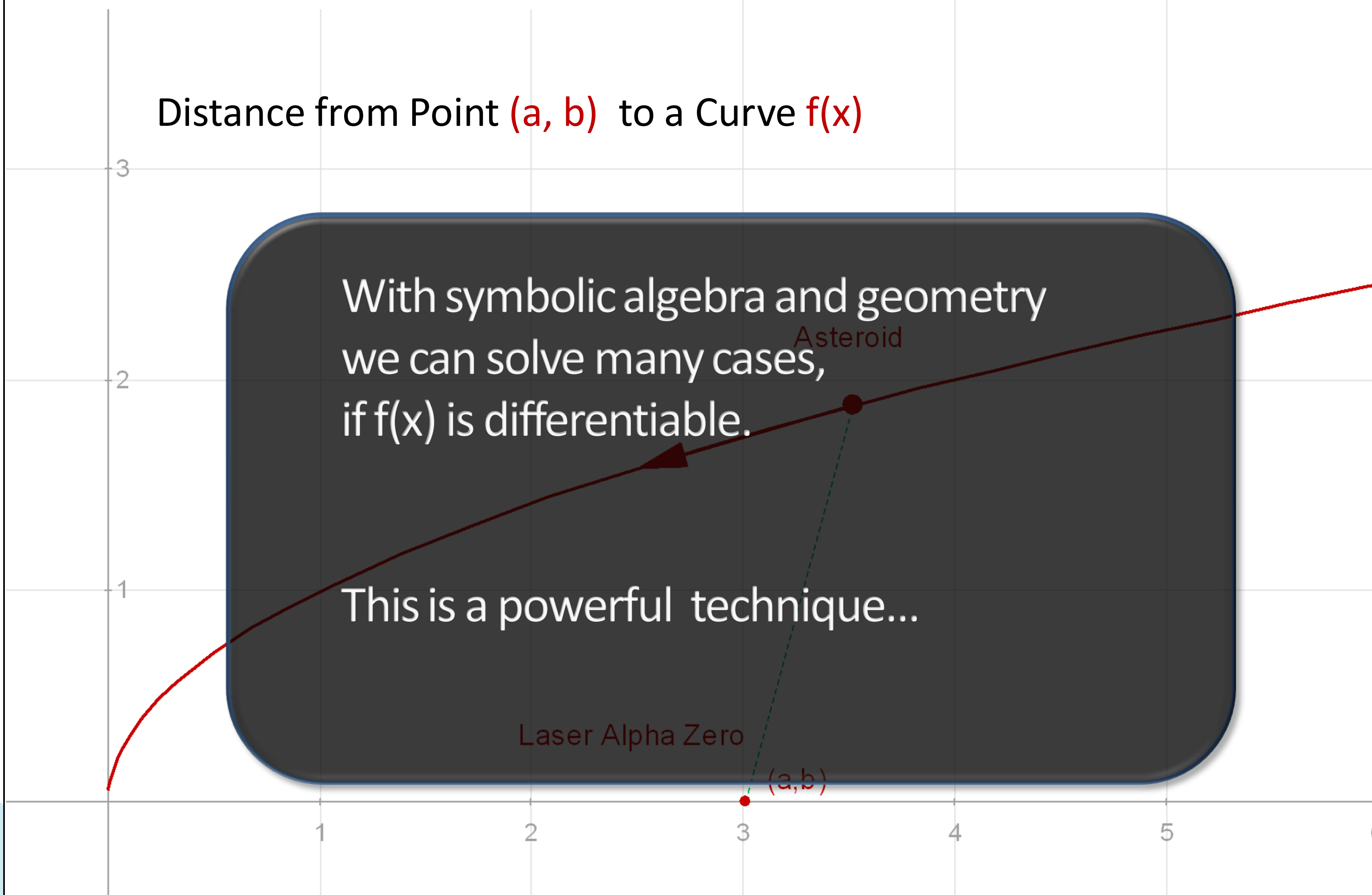
Distance from Point (a, b) to a Curve f(x)



Asteroid Closest Approach:

Lecture18-ClosestApproach.gx

Distance from Point (a, b) to a Curve f(x)



With symbolic algebra and geometry
we can solve many cases,
if f(x) is differentiable.

This is a powerful technique…

Asteroid Closest Approach:

Lecture18-ClosestApproach.wxm

```
distance(x,y,a,b):=sqrt( (x-a)^2 + (y-b)^2 );
```

$$\mathrm{distance}(x, y, a, b) := \sqrt{(x-a)^2 + (y-b)^2}$$

```
distance(r,s,t,u);
```

$$\sqrt{(s-u)^2 + (r-t)^2}$$

```
distance(1,1,2,2);
```

$$\sqrt{2}$$

```
distance(x,f(x),a,b);
```

$$\sqrt{(f(x)-b)^2 + (x-a)^2}$$

- Define a distance function.
- Test it symbolically.
- Test it numerically.
- Test it functionally.

Asteroid Closest Approach:

Lecture18-ClosestApproach.wxm

- Specify trajectory f(x).
- Diff. distance function.
- Find x of min. distance.
- Back substitute to find min. y.

```
f(x):=sqrt(x);
```

$$f(x):=\sqrt{x}$$

```
closest:diff(distance(x,f(x),3,0),x);
```

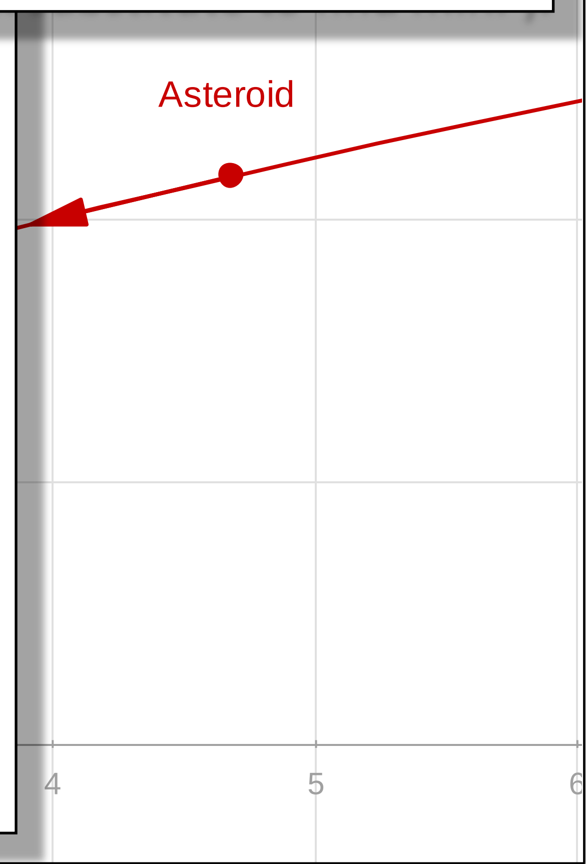
$$\frac{2(x-3)+1}{2\sqrt{x+(x-3)^2}}$$

```
xClosest : rhs(first(solve(%,x)));
```

$$\frac{5}{2}$$

```
yClosest : f(xClosest);
```

$$\frac{\sqrt{5}}{\sqrt{2}}$$



Asteroid Closest Approach:

Lecture18-ClosestApproach.wxm

```
closestApproach:distance(xClosest,yClosest,3,0);
```

$$\frac{\sqrt{11}}{2}$$

```
fullVector:[xClosest, yClosest, closestApproach];
```

$$[\frac{5}{2},\frac{\sqrt{5}}{\sqrt{2}},\frac{\sqrt{11}}{2}]$$

```
float(fullVector);
[2.5,1.58113883008419,1.658312395177]
```

- Compute closest distance.
- Package results.
- Compute decimal form.

⇒ ~1.658

Exercises:

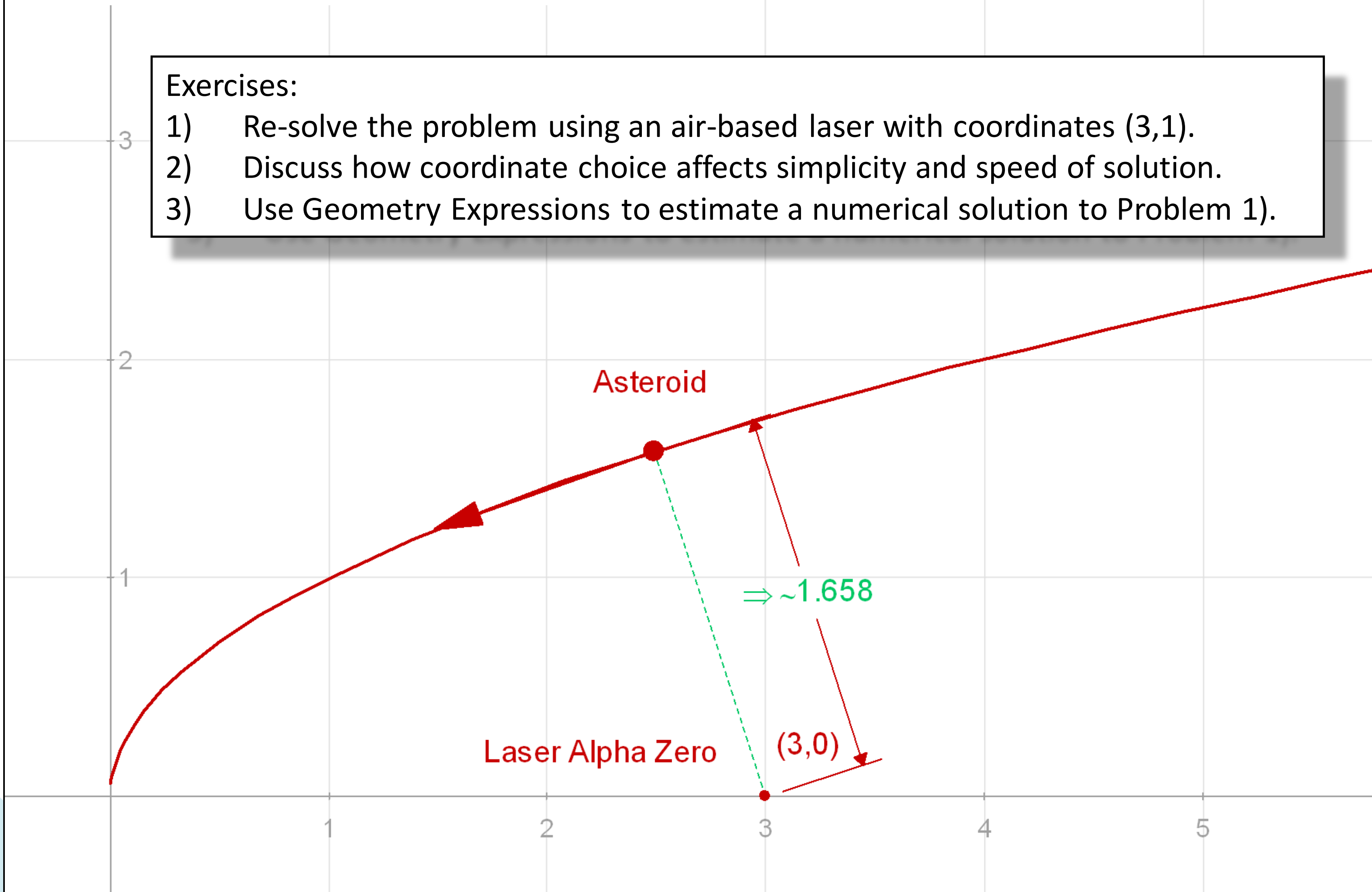
1) Which version of the fullVector is most suitable for transmission over a noisy communication channel? Why?

Asteroid Closest Approach:

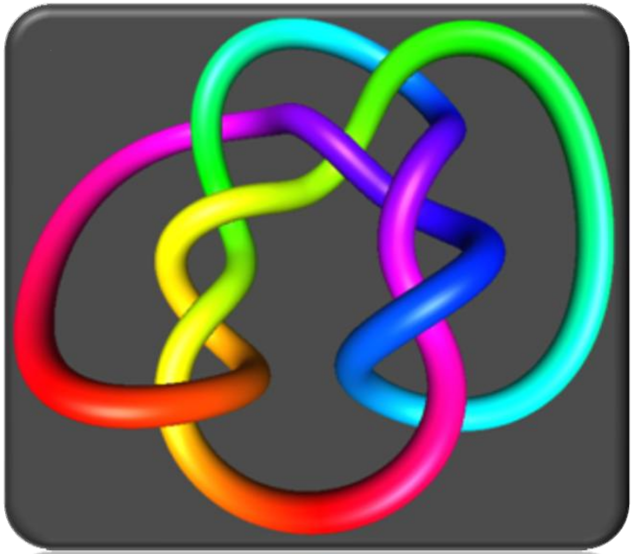
Lecture18-ClosestApproach.gx

Exercises:

1) Re-solve the problem using an air-based laser with coordinates (3,1).
2) Discuss how coordinate choice affects simplicity and speed of solution.
3) Use Geometry Expressions to estimate a numerical solution to Problem 1).



End



- Knot rendered using KnotPlot™