SOME APPLICATIONS OF FIRST-ORDER EQUATIONS.

1. *Shape of a liquid in a spinning tank.* A cylindrical tank half-filled with liquid turns about its axis with uniform angular velocity ω (rad/s). Show that the profile of the free liquid surface is parabolic.

Solution. A particle of liquid (of mass m) on the surface is in equilibrium (doesn't move up or down) under the action of two forces: the centripetal force (radial outwards, magnitude $m\omega^2 x$, where x is the distance from the axis) and its weight (pointing down, magnitude mg). The resultant of these two forces must be normal to the free surface. Drawing a diagram, if $dy/dx = tan(\theta)$ is the slope of the cross-section $y(x)$ at distance x we see this implies: $\tan(\theta) = m\omega^2 x/mg$, or $dy/dx = (\omega^2/g)x$, with solution $y(x) = (\omega^2/2g)x^2 + C$, a parabola.

2. *The airplane problem.* An airplane flies towards a town located due west of the starting airport, at a distance a. The airplane's speed (relative to the air) is v, and the pilot keeps the plane oriented towards the town at all times. However, a wind blows from the south, with speed w (rel. to the ground). Find an equation for the path of the airplane. Will it reach its destination?

Solution. The velocity vector of the plane relative to the ground is the resultant of wind velocity and velocity rel. to air, and has components:

$$x'(t) = -v\cos\theta, \quad y'(t) = -v\sin\theta + w$$

where $\tan\theta = y/x$ when the plane is at (x, y) (the origin placed at the destination, the x-axis going through the starting city.) Since $\sin\theta = y/\sqrt{x^2+y^2}$, $\cos\theta = x/\sqrt{x^2+y^2}$, dividing the two we obtain:

$$dy/dx = y/x - k\sqrt{1 + y^2/x^2}, \quad (k = w/v)$$

an equation of 'homogeneous type'. Changing to $u = y/x$, we obtain:

$$x\frac{du}{dx} = -k\sqrt{1+u^2}, \quad u(a) = y/a = 0$$

and the solution is:

$$y(x) = (x/2)[(x/a)^{-k} - (x/a)^k].$$

Interpretation: There are three cases: (i) $k = 1$: the path is a parabola; (ii) $k > 1$: then $\lim_{x\to 0} y(x) = \infty$. In these two cases, the plane never reaches the town. (iii) $k < 1$: then $\lim_{x\to 0} y(x) = 0$, and the plane reaches its destination.

3. *The shape of a suspension cable.* A suspension cable hangs from two suspension points, acquiring a curved shape under the influence of its own weight. Find an equation for the curve describing this shape.

Solution. Let $y(x)$ describe the shape, with the axes chosen so that $y'(0) = 0$. Consider a section of cable between $P(x, y(x))$ and $P_1 = (x_1, y(x_1),$ where $x_1 > x$, in equilibrium under the action of three forces: the tension $T(P_1)$, the tension $T(P)$ (both tangent vectors to the curve, pointing right and left resp.) and the weight $\rho g \Delta s$, where ρ is the linear mass density and Δs the length of cable from P to P_1. With θ resp. θ_1 the tangent angles at x and x_1, the balance of forces gives:

$$T\cos\theta = T_1\cos\theta_1$$

$$T_1\sin\theta_1 = T\sin\theta + \rho g\Delta s,$$

so $T\cos\theta$ is constant along the curve, call it H. Since $\tan\theta = dy/dx$, we obtain:

$$H(\frac{dy}{dx}(x_1) - \frac{dy}{dx}(x)) = \rho g \Delta s,$$

so letting $x_1 \to x$ and recalling the expression for ds, the element of arc length of a graph, we find the equation:

$$H\frac{d^2y}{dx^2} = \rho g\sqrt{1 + (dy/dx)^2}.$$

This is a first-order equation for $z(x) := dy/dx$; letting $\rho g/H = k$:

$$\frac{dz}{dx} = k\sqrt{1+z^2}, \quad z_{|x=0} = 0,$$

with solution:

$$z + \sqrt{1+z^2} = e^{kx},$$

or equivalently $z = \sinh(kx)$. Integrating, we find:

$$y(x) = C + \frac{1}{k}\cosh(kx),$$

the *catenary.*